52 e 7

Geometry made Eaſy;

OR,

A New and Methodical Explanation

OF THE

ELEMENTS of GEOMETRY.

CONTAINING,

I. A very eaſy and conciſe Commentary on the firſt SIX, XI, XII, XIII, XIV, and XV Books of EUCLID, and the moſt material Propoſitions of ARCHIMEDES, concerning the CIRCLE, and its Quadrature, the CYLINDER, CONE, and SPHERE.

II. A compendious Treatiſe of ALGEBRA, with its Application in the Solution of ſeveral curious and uſeful GEOMETRICAL PROBLEMS.

III. A Collection of RECREATIVE PROBLEMS, propoſed for the Learner's Diverſion, being chiefly extracted from OZANAM's Mathematical Recreations.

IV. An Introduction to CONIC SECTIONS, containing a familiar Explanation of the moſt principal Properties of the ELLIPSIS, PARABOLA, HYPERBOLA, &c.

To which is added,

An entire new, curious and exact Method of exhibiting in Miniature, the various kinds of Solids, REGULAR and IRREGULAR, and alſo their SECTIONS; each being diſtinctly and exactly ſhewn as they really are in their natural State, by Schemes cut out of PASTE-BOARD: by which means the Doctrine of Solids will be much eaſier comprehended than by any other Method yet Publiſhed.

By JOHN LODGE COWLEY,

Late Maſter of the ACADEMY in St. Martin's-Lane.

Recommended and approved by ſeveral very eminent MATHEMATICIANS, as the moſt proper Book on this Subject, for the Uſe of MATHEMATICAL SCHOOLS, and ſuch as would learn the Principles of this Science by their own Application only.

LONDON:
Printed for the AUTHOR, and Sold by Mr. Watkins, Charing-Croſs;
Mr. Ayſcough, in Ludgate-Street, Opticians; Mr. Heath, in the
Strand; Mr. Bennet, in Crown-Court, St. Ann's, Soho, Mathematical Inſtrument-Makers; Mr. Roberts, Engraver, Holborn.

Mathematics and Accompts taught by the AUTHOR.

G III R
BRITISH MUSEUM

G III R
HONI SOIT QUI MAL Y PENSE

PREFACE.

THE *Excellency and Uſefulneſs of the Science, which is the Subject of the following Sheets, have been for many Ages too well known and experienced to need any Encomium at this Time of Day; and indeed an Attempt to ſhew the great Advantages, which Mankind hath reap'd from the Knowledge of the* Mathematics, *were as ridiculous as to endeavour the exact* Quadrature of the Circle, *or the inveſtigating the* true Root of a Surd Number. *I ſhall therefore only beg leave to acquaint the Reader, that my principal View in ſending abroad this Work, which contains the firſt* Rudiments of Geometry, &c. *was to introduce him to the Acquaintance of thoſe learned Authors who have entered more deeply into this Science, and to render it fit for the Uſe of* Mathematical Seminaries, *or of ſuch as would attain to the Knowledge of* Geometry *by their own Application. Perſpicuity and Brevity are therefore the Ends here chiefly aim'd at; in order to which, I have retrench'd or added where I found my Author too ſuperfluous, or obſcurely conciſe.*

And for the better underſtanding the various Kinds of Solids, *which are ſo perplexing to young Minds, and frequently deter 'em from the Proſecution of theſe Studies, I have added an entire new Method of repreſenting them and their ſeveral Sections on Paſte Board, ſo as to exhibit at one View their different Forms and Proportions; by which Means the Difficulties which have hitherto attended the Contemplation of thoſe Bodies, when expreſſed by Lines, are entirely obviated, and the Doctrine of Solids render'd perfectly plain and eaſy,* *without any previous Knowledge in the* Principles of Perſpective.

Altho' it be not the Province of this Treatiſe, as a Commentary on the Elements of Euclid, *to treat of more than the Five* regular Solids, *yet as thoſe Bodies may be transformed into many irregular ones; I thought it might not be a uſeleſs Curioſity, to preſent the Reader with a Delineation of the Plat forms of ſome few of them, according to this new Method.*

Neither the Price nor the Bounds which I preſcribed to this Book when I firſt publiſh'd my Propoſals for a Subſcription, would permit me to ſwell it with any farther Additions; but if what is here offer'd for explaining the Doctrine of Solids contain'd in Euclid's XI and XII Books, be approv'd, I ſhall hereafter, as I have Leiſure and Opportunity, extend it to his other Propoſitions.

It may perhaps be ask'd what ſhould induce me to encreaſe the Number of Books which are already Extant on this Subject, and more eſpecially as ſome very eminent Mathematicians have ſo warmly inveigh'd againſt all Performances of this Nature, and aſſerted Euclid's Method of ranging and demonſtrating the Propoſitions to be better than any other? &c. &c. *To this I ſhall only anſwer, that in my own Experience the Knowledge of Geometry hath by the following Method been eaſily attain'd by thoſe to whom Euclid's prov'd very difficult; and this in the Judgment of the impartial Reader will, no doubt, ſufficiently juſtify me in this Attempt.*

London, July 23,
1752.

J. L. C.

ADVERTISEMENT.

THE irregular Solids ſuper-added to this Work, may be had cut thro' compleatly, at the additional Expence of *One Shilling and Sixpence.* The Paſte Board Plates may be had alone compleatly cut. Price *Six Shillings*. And if cut only as belonging to the Book, Price *Four Shillings and Sixpence*.

Taught by the Author.

The ſeveral Parts of Mathematics and Accounts, requiſite to qualify Youth for either the

ARMY,
SEA, PUBLIC-OFFICES,
COUNTING-HOUSE,
OR,
TRADE,

In the moſt eaſy and accurate Method,

At his OFFICE,
in Middle-Scotland-yard.

ALSO

Gentlemen and Ladies *taught Abroad.*

A TABLE

Shewing how to find the Propoſitions of Euclid treated of in this Work.

The firſt Figure denotes the Propoſition of Euclid; and the ſecond, which is a Roman Figure, alludes to the Book; and the third, in what Number thereof, it is contained in this Treatiſe.

EUCLID, Book I.

1	II.	69	14	II.	20	26	II.	95	37	II.	134
2	I.	30	15	II.	21	27	II.	26	38	II.	134
3	I.	30	16	II.	74	28	II.	28	39	II.	136
4	II.	98	17	II.	79	29	II.	25	40	II.	136
5	II.	83	18	II.	89	30	I.	73	41	II.	133
6	II.	90	19	II.	89	31	I.	72	42	II.	138
7	II.	95	20	II.	66	32	II.	79	43	II.	132
8	II.	94	21	I.	57			73 & 75	44	II.	132
9	II.	31	22	II.	67	33	II.	112	45	II.	140
10	I.	48			& 68			& 117	46	II.	121
11	I.	47	23	II.	29	34	II.	128	47	II.	142
12	I.	46	24	II.	104	35	II.	129		IV.	78
13	II.	17	25	II.	104	36	II.	129	48	II.	142

EUCLID, Book II.

1	III.	17	5	III.	21	9	III.	25	13	III.	28
2	III.	18	6	III.	22	10	III.	26	14	IV.	84
3	III.	19	7	III.	23	11	IV.	54			
4	III.	20	8	III.	24	12	III.	27			

EUCLID, Book III.

1	I.	87	10	I.	89	18	I.	106	29	I.	52
2	I.	83	11	I.	81	19	I.	108	30	I.	85
3	I.	91 & 92	12	I.	82	20	II.	41	31	II.	44
4	I.	93	13	I.	80	21	II.	40			45
	II.	48	14	I.	94	22	II.	114			& 48
5	I.	17	15	I.	95	23	I.	98	32	II.	67
6	I.	78			& 96	24	I.	31	33	II.	49
7	I.	102	16	I.	106	25	I.	80	34	II.	48
8	I.	99			& 109	26	II.	42	35	IV.	55
9	I.	102		II.	32	27	II.	43	36	IV.	56
		& 104	17	I.	113	28	I.	32	37	IV.	59

EUCLID, Book. IV.

1	I.	30	6	II.	124			& 151	15	IV.	143
2	II.	100	7	II.	126	11	IV.	158	16	IV.	163
3	II.	101	8	II.	125	12	IV.	160			
4	II.	102	9	II.	127	13	IV.	161			
5	II.	70	10	IV.	149	14	IV.	162			

EUCLID, Book V.

1	III.	64	8	III.	91	15	III.	64	20	III.	95
2	III.	65	9	III.	52	16	III.	46	21	III.	96
3	III.	66	10	III.	92			& 56	22	III.	62
4	III.	67	11	III.	53	17	III.	51	23	III.	88
5	III.	68	12	III.	50			& 57	24	III.	63
6	III.	69	13	III.	93	18	III.	49	25	III.	97
7	III.	70	14	III.	94	19	III.	48			

EUCLID, Book VI.

1	III.	99	10	IV.	20			& 85	27	IV.	94
2	IV.	16	11	IV.	23	19	IV.	71	28	IV.	95
3	IV.	17	12	IV.	24	20	IV.	90	29	IV.	96
4	IV.	10	13	IV.	29	21	IV.	11	30	IV.	34
5	IV.	12	14	IV.	82	22	IV.	74	31	IV.	79
6	IV.	13	15	IV.	83	23	IV.	81	32	IV.	15
7	IV.	14	16	III.	56	24	IV.	51	33	IV.	46
8	IV.	27	17	III.	57	25	IV.	86			
9	IV.	22	18	IV.	26	26	IV.	52			

EUCLID, Book. XI.

1	V.	7	5	V.	21	9	V.	35	13	V.	23
2	V.	8	6	V.	29	10	V.	36	14	V.	33
3	V.	10	7	V.	14	11	V.	18	15	V.	37
4	V.	19	8	V.	29	12	V.	22	16	V.	34

17	V.	38	23	V.	52	29	V.	109	35	V.	54
18	V.	31	24	V.	107	30			36	V.	138
19	V.	30	25	V.	108	31			37	V.	139
20	V.	49	26	V.	53	32	V.	111	38	V.	35
21	V.	51	27	V.	103	33	V.	131	39	V.	106
22	V.	50	28	V.	105	34	V.	136	40	V.	114

EUCLID, Book XII.

1	IV.	92	7	V.	119	12	V.	134	17	V.	147
2	IV.	93	8	V.	133	13	V.	113	18	V.	148
3	V.	124	9	V.	137	14	V.	112			
4	V.	125	10	V.	128			& 130			
5	V.	123	11	V.	112	15	V.	137			
6					& 130	16	V.	146			

EUCLID, Book XIII.

1	IV.	66	7	IV.	168	13	V.	150			& 167
2	IV.	68	8	IV.	169			& 152	17	V.	161
3	IV.	69	9	IV.	154	14	V.	156			& 162
4	IV.	70	10	IV.	164	15	V.	153	18	V.	173
5	IV.	35	11	IV.	170			& 154			
6	IV.	139	12	IV.	146	16	V.	171			

EUCLID, Book XIV.

1	V.	176	3	V.	172	5	V.	178	7	V.	180
2	IV.	67	4	V.	174	6	V.	179	8	V.	159

EUCLID, Book XV.

1	V.	181	3	V.	183	4	V.	184	5	V.	185
2	V.	182									

A
LIST
OF
SUBSCRIBERS.

A.

MR. James Ayſcough, *Optician*, Ludgate-ſtreet, 2 Books.

B.

The Right Rev. Lord Biſhop of Briſtol.
William Bullock, Eſq;
John Bagnall, Eſq;
George Barber, Gent.
Rev. Mr. Blacow, A. M.
Mr. Archibald Bontein, *Engineer*.
Mr. Charles Brent, *Mathematician*, 3 Books.
Mr. Francis Broad, *Writing-Maſter and Accomptant*, New Belton-Street.
Mr. John Bennet, *Mathematical Inſtrument-Maker*, Crown Court, St. Ann's, Soho.
Mr. George Bickham, *Engraver*, May's Buildings, 2 Books.
Mr. Boydell.
Mr. Francis Balf.
Mr. Bandy, *Carver*.
Mr. Buſh.

Mr. John Buttcher, *Joiner.*
Blue-Poſts Lodge, Old Bond-Street. ∟

C.

Mr. Edward Crouch, *Joiner.*
Mr. Thomas Cleeter, *Carpenter.*

D.

Mr. William Davis, *Attorney*, Thavies-Inn.
Mr. Drew, *Vintner.*
Mr. Dunkeley, *Baker.*
Mr. John Deſborough, *Carpenter*, Duke-Street, Groſvenor-Square.

E.

Mr. Ellis, *Chairmaker*, St. Ann's, Soho.
Mr. Egreman, *Surveyor*, Brownlow-Street, Long-Acre.
Mr. Ellicott.
Mr. James Elderton, *Joiner.*
Mr. Edringham.

F.

Mr. Freeman, *Gunſmith*, Strand.
Mr. Fletcher, *School-maſter*, Oxford-Market.
Mr. John Fearnſide, *Stationer*, Wiſbech, in the Iſle of Ely.
Mr. William Field, *Carpenter.*
Fiſh and Bell Lodge, Soho-Square. ∟

G.

John Greenhill, Eſq; Lincolns-Inn, 2 Books.
Mr. John Griffith, *Chemiſt*, Hay-Market.
Mr. Henry Groat, Duke's-Place, St. James's, 2 Books.
Mr. Samuel Gribble, *Painter.*

Mr. Thomas Gunan, *Harpſichord-Maker*, May-Fair.

Mr. James Gordon, *Carpenter*, St. Martin's-Lane.

George Lodge, Grafton Street. L

H.

Abraham Hume, Eſq;

———Horner, Eſq;

Miſs Peter Hilliard.

Mr. Alexander Hume.

Mr. Heath, *Mathematical Inſtrument-Maker*, Strand.

Mr. Samuel Healer, *Writing-Maſter*.

Mr. Haynes, *Engraver*, Charing-Croſs.

Mr. Hazard.

Mr. Hariſſon.

Mr. Hodgkins.

Mr. Andrew Henderſon, *Bookſeller*.

Mr. Hearne, *Watch-maker*.

J.

Rev. Mr. James.

Mr. Jump, *Carpenter*.

Mr. Henry Jones, little Pulteney-Street.

Mr. Johnſon, *Mariner*.

Mr. Evan Jones, *Taylor*, Strand.

K.

The Rev. Mr. Brewer Kidman, A. B.

King's-Head Lodge, Holborn. L

L.

The Rt. Hon. Lord Viſcount Liſburne.

Thomas Lediard, Eſq;

———Lewis, M. D. Kingſton upon Thames.

Mr. Howell Lewis, *Surgeon*, Jermyn-Street, St. James's.

Mr.

A LIST of SUBSCRIBERS.

Mr. Lindſay, *Watch-Maker* to His Royal Highneſs the Prince of Wales, Strand.
Mr. William Lyon, *Joiner*.
Mr. Luke Lewin, *Bricklayer*.
Mr. J. Lewis.
Mr. Lacey, *Bookbinder*, in St. Martin's-Lane.
Red-Lyon Lodge, York-Street, St. James's-Square. ∟

M.

The Hon. Andrew Mittchell, Eſq; *Member of Parliament for Aberdeen*.
The Hon. Capt. Montague.
Richard Matthiſon, Eſq;
George Mackenzie Quin, Eſq;
Quin Mackenzie Quin, Eſq;
Daniel Malpas, Eſq;
Mr. Matthews, *Watch-Maker*, Saliſbury-Court, Fleet-Street.
Mr. John Morris.
Mr. Thomas Malyn.
Mr. John Moncrieffe, *Church-Court*, Strand.
Mr. David Morris.
Mr. Peter Marriott, *Sculptor*.
Mr. Marſhall, *Glazier*, King-Street, Seven Dials.
Mourning-Buſh Lodge, Alderſgate. ∟

N.

The Rt. Rev. Lord Biſhop of Norwich.
Mr. Lawrence Netter.
Mr. Richard Neilſon.

O.

Mr. Oliver, *Teacker of Mathematicks*, Greenwich, 4 Books.
Mr. Oram, *Maſter-Carpenter* to the Board of Works.
Mr. Oliver, *Button-Maker*.

P.

P.

Mr. James Praed, *Attorney*.
Mr. Thomas Palmer.
Mr. James Pitts, *Carpenter*.
Mr. Philip Pattenden, *Carpenter*.
Mr. Pepys, *Book-binder*, King's-Head-Court, Holborn.

R.

Mr. John Rocque, *Chorographer* to His Royal Highneſs the Prince of Wales, 6 Books.
Mr. John Rule, A. M. *Maſter of the Academy*, Wapping.
Capt. John Rickard, Camelford, Cornwall.
Mr. Robins, Holborn.
Mr. Bateman Robſon.
Mr. John Robinſon, *Carpenter*.
Mr. John Richardſon.
Mr. Ridley, *Bookſeller*, Charing-Croſs.

S.

The Rt. Hon. the Counteſs Dowager of Stafford.
The Hon. Walter Scott, Eſq; *Member of Parliament for the Shire of Tiverdale.*
Mr. Thomas Shepheard, *Attorney*, } Great Ruſſel-Street, Bloomſbury.
Mrs. Elizabeth Shepheard, }
Mr. John Sherwin, *Surgeon*, Deptford.
Mr. Scarlet, *Painter*, Brownlow-Street, Long-Acre, 2 Books.
Mr. Smith, *Writing-Maſter and Accomptant*, at March, in the Iſle of Ely.
Mr. Short, *Teleſcope-Maker*.
Mr. Obadiah Shadd.
Mr. William Sparling, *Carpenter*.

A LIST of SUBSCRIBERS.

T.

Mr. Thomas, *Upholder*, Oxford-Market.
Mr Mark Tideman, *Joiner*.
Mr. John Taylor, *Joiner*.
Mr. Tyler, *Surveyor*.
Mr. Henry Therond.

V.

Mr. John Vardy, *Clerk* to the Board of Works, Scotland-Yard.

W.

The Rt. Hon. Lady Betty Warren.
John Warden, Eſq;
Mr. Watkins, *Optician*, Charing-Croſs.
Mr. John Weſtley, *Coach-Maker*, Pall-Mall.
Mr. Walkingham.
Mr. Joſhua Williams, *Accomptant*, Briſtol.
Mr. Henry White, *Bricklayer*, Oxendon-Street.
Mr. William Whitehouſe.
Mr. Ward, *Carpenter*.
Mr. Thomas Winefreed, *Joiner*.
Mr. William Wetherilt, *Painter*, Short's-Gardens, Holborn.
Mr. Lawrence Whyte.
Mr. Wells, *Carpenter*, Hoſier-Lane, Smithfield.

THE
CONTENTS.

BOOK I.

Page
SECTION I. OF *the different* Species *of Extenſion.* 1
SECT. II. Of *right Lines.* 7
SECT. III. Of *the Circular Line.* 8
SECT. IV. Of *the different Poſition of two right Lines with reſpect one to another.* 10
SECT. V. Of *the different Poſition of two Circles with regard to each other.* 17
SECT. VI. Of *the Poſition of a right Line with reſpect to a Circle.* 19

BOOK II.

SECTION I. OF *Angles.* 27
SECT. II. Of *the Compariſon of Angles, and of their different Poſitions with reſpect to a Circle.* 33
SECT. III. Of *Triangles.* 39
SECT. IV. Of *Multilaterals, or Figures having ſeveral ſides.* 47
SECT. V. Of *the Area of Surfaces.* 52

BOOK III.

SECTION I. OF *Addition, Subtraction, &c. upon Lines, Planes and Solids.* 60
SECT. II. Of *the Power of Lines.* 64
SECT. III. Of *the Ratio's and Proportions of Lines, Surfaces and Solids.* 70

SECT. IV. Of *Compound Ratio's and their Properties.* 80
SECT. V. Of *the Compariſon of Ratio's.* 83

BOOK IV.

SECT. I. OF *the Ratio's and Proportions of Lines.* 88
SECT. II. Of *the Ratio's and Proportions which the Sides of Triangles bear to each other.* 91
SECT. III. Of *the Ratio's and Proportions which the Circumferences of two or more Figures bear to each other, and to the Radii of their circumſcribing Circles.* 100
SECT. IV. Of *the Ratio's and Proportions of Surfaces.* 103
SECT. V. Of *the Commenſurability, or Incommenſurability of Lines and Surfaces.* 117
SECT. VI. Of *the Ratio's between the Chords and Radii of the Circle.* 125
Of *the Quadrature of the Circle.* 133

BOOK V.

SECT. I. OF *the Sections and meeting of Planes.* 138
SECT. II. Of *the Compoſition of Solids according to their Surfaces, and Solidity.* 145
SECT. III. Of *the Surfaces of Solids.* 151
SECT. IV. Of *the Solidity of Solids.* 159
SECT. V. Of *the Inſcribing or Circumſcribing a Sphere with the five regular Bodies.* 168
Recreative Problems. 180

BOOK VI.

CHAP. I. OF *the Examination of a Queſtion, or Problem. In the firſt Place, it muſt be well underſtood, and exactly expreſſed.* 185
CHAP. II. *Altho' the Lines and Quantities concerned in a Queſtion are unknown, yet they may be* ex-

expreſſed, and all the Operations of Arithmetic performed upon them. 186
Chap. III. *After having expreſſed a Queſtion or Problem, and deſcribed the Figure proper thereto, &c.* 189
Chap. IV. *The Proportions which are between the Lines of the Figure by which a Problem is expreſſed, &c.* 190
Chap. V. *There muſt be found as many Equations as there are unknown Lines, and all theſe Equations muſt be reduced to one only.* 191
Chap. VI. *The Terms of an Equation muſt be reduced to the ſimpleſt Expreſſion, &c.* 192
Chap. VII. *Equations are of one or more Dimenſions or Degrees, and by thoſe Degrees it is that Problems are diſtinguiſhed.* 195
Chap. VIII. *Of the Geometrical Conſtruction or Effection of Equations, &c.* 196
Chap. IX. *Of the Geometrical Conſtruction of a Locus, or Place.* 199
Chap. X. *Solid Problems are, &c.* 202
Chap. XI. *The Solution of a Problem, &c.* 202

Introduction to Conic Sections.

Chap. I. *Of the Curve Lines which repreſent the different Sections of a Cone, &c.* 209
Chap. II. *Of the Parabola, or Curve Line which repreſents the Section of a right Cone, &c.* 211
Chap. III. *Of the Ellipſis, &c.* 214
Chap. IV. *Of the Hyperbola, &c.* 219

E R R A T A.

THE marginal Numbers from 28 *in Page* 8, *to* 52 *in p.* 13, *ſhould each be accounted an Unit more,* 28 *ſhould be* 29, *and ſo on.* *P.* 13, *l.* 7, *f.* EA, *r.* ED. *P.* 16, *numb.* 71, *l.* 5, *f.* CD upon D, *r.* CD upon Z. *P.* 20, *l.* 9, *f.* D, *r.* B. *P.* 21, *l.* 18, *f.* BD, *r.* CD. *P.* 23, *l.* 9, *make a* full-ſtop at BD. *P.* 24, *l.* 3, *leave out*, as CD. FG. &c. *P.* 30, *numb.* 21, *l.* 8, *f.* EAD, *r.* EAB. *P.* 31, *l.* 4, *f.* BE, *r.* BF. *Ibid, l.* 20, *f.* ED, *r.* BD. *Ib. l.* 35, *f.* GBF, *r.* GEF *P.* 42, *l.* 11, *f.* an Iſocles, *r.* two Iſocles, *and eraſe the Word of, in the l. above.* *P.* 48, *l.* 9, *f.* DEF, *r.* DFE. *Ib. num.* 116, *l.* 5, *f.* BC, *r.* AB. *P.* 61, *l.* 33, *f.* ABCD×E, *r.* ABCD×DE. *P.* 63, *rule* 2, *l.* 7, *remove the Word* by, *and put it next the Word* much. *P.* 64, *num.* 11, *l.* 8, *next to the Word* makes, *read as follows*; only 6, whereas 3 times 3, or 3 multiplied by itſelf, makes. *P.* 67, *Prop.* 6, *l.* 5, *put the Word* half, *next the Word* of. *Ib. l.* 14, *f.* BC, *r.* $\overline{BC}^2$ *P.* 68, *l.* 10, *r.* $2\,AB\times BC=2bd+2dd$. *next l. r.* $2AB\times BC$, &c. *next l. r.* $\overline{AB}^2+\overline{BC}^2$. *Ib. l.* 24, *after the Word* adding, *r.* $\overline{AC}^2$ *to one ſide and its equal bb to the other.* *P.* 86, *l.* 34, *f.* A+C, *r.* A+D. *P.* 89, *l.* 20, *f. the part* B. *r.* BH. *P.* 92, *f.* E and F, *r.* C and F. *P.* 103, *num.* 51. *l.* 5, *f.* CEFK, *r.* CEFH. *P.* 105, *l.* 1, *f.* CE, *r.* CD. *P.* 106, *l. the laſt, for* BD×AE×BD, *r.* BD×AE+BD. *P.* 108, *l.* 8 *f.* BG, *r.* EG. *P.* 113, *l.* 28 *f.* GAI, *r.* GHI. *Ib. l.* 30, *f.* ABCD, *r.* ABCE. *P.* 143, *l.* 4, *f.* ED, *r.* BD. *Ib. l.* 7, *f.* AB, *r.* AP. *P.* 155, *l. the laſt but three, f.* GM, *r.* CM. *P.* 156, *l.* 9, *f.* BG. *r.* BO. *P.* 162, *l.* 38, *f.* ADE GLE, *r.* ADEGLF. *P.* 164, *laſt l. but two, f.* theſe two numbers, *r.* three numbers. *P.* 170, *l.* 18 *near its end, f.* nn×2ss, *r.* nn=2ss. *P.* 174, *l.* 14, *f.* Y, *r.* X. *Ib. l.* 3-, *f.* ICF, *r.* IGF. *P.* 1;8, *l.* 27, *f.* ×, *r.* +. *Ib. l. the laſt, f.* ×, *r.* +. *P.* 181, *l.* 42, *f.* EH, *r.* FH. *P.* 188, *laſt line but three, next to* GH, *r. by* FG. *P.* 203, *l.* 38, *f.* x—, *r.* x—b. *P.* 205, *l. the laſt, f.* ac+d, *r.* ac+dx. *P.* 206, *l.* 15, *f.* c—, *r.* c—a. *P.* 212, *l.* 11, *f.* $\overline{EP}^2$, *r.* $\overline{FP}^2$. *P.* 213, *l.* 18, *f.* EMP, *r.* FMP.

AN EXPLANATION OF THE TERMS and CHARACTERS Uſed in this TREATISE.

AXIOM.

IS a ſelf evident and immutable Truth, perceived and conſented to by every one at firſt ſight.

DEMAND.

Is a Propoſition which doth not appear Self-evident at firſt ſight, but becomes ſo ſoon as it is conſidered, and is therefore juſtly required to be granted as undeniable.

DEFINITION.

Is a Propoſition that explains the meaning of a Word; or which gives an exact Idea of the thing Signified by that Word.

THEOREM.

Is likewiſe called a Propoſition, the truth of which muſt be Demonſtrated.

PROBLEM.

Is alſo a Propoſition which muſt be demonſtrated, but there is in it ſomething required to be done; and that which was required to do, muſt be proved that it is done.

LEMMA.

Is a Propoſition made uſe of, in order to abbreviate the Proof of the following Demonſtration.

CORALLARY.

Is a truth or conſequent Propoſition, gained from ſome preceeding Demonſtration.

DEMONSTRATION.

Is a train of arguments, deduced from ſuch plain Axioms and other Self-evident Truths, as cannot be denied by any One that conſiders them, but convince the Mind of the truth thereof, even of the moſt obſtinate, who will grant nothing but what is extorted by its undeniableneſs, and is the higheſt degree of Proof, human Reaſon is capable of attaining to.

SCHOLIUM.

Is a brief commentary, or Obſervation upon ſome precedent Diſcourſe.

N. B. Euclid includes the Terms *Theorem* and *Problem* both in one general Term *Propoſition*.

CHARACTERS.

This mark + ſignifies *plus*, or *more*, $A+B$, that is A more B, This — Signifies *Minus* or *leſs*. $A-B$, Signifies A leſs B. = is the Mark or Sign of *Equality*. $C=D$, ſignifies that C is *equal* to D.

> *Greater*, < *Leſſer*.

X *By*, This is the Sign of Multiplication, as $A\times D$ ſignifies A multiplied by D.

Sup. Supra or *above*. b. *Book*. n. *Number*.

There are figures put in the Margin of this Book, which ſhew how to find the Propoſitions alluded to, thus b. 2. n. 6. ſignifies *Book the Second, Number the Sixth*. But if the Place referred to, is in the ſame Book, then it is cited with this note *ſup*. thus *ſup*. n. 5. —that is *number the fifth above*.

When the Propoſition treated of is to be found in Euclid the Place is denoted thus: *Eucl.* 1. *Prop.* 7. that is, Euclid Book the Firſt, *Propoſition the Seventh*.

As to what other Notes may be uſed in this Work, they will be explained in the Places where they firſt occur; and that the Uſe of the above Characters may become Familiar, I ſhall make uſe of them in propoſing the following Truths.

AXIOMS.

1. The *whole is greater than its part*.

If A and B are the parts of the Line X, the whole Line X is greater than either A or B taken ſeparately.

2. The *Whole is equal to all its parts taken together*.

If A and B are all the parts of X, it is evident that $A+B$, that is A with B is equal to X: which is expreſſed thus, $A+B=X$.

3. QUANTITIES *which are Equal to the ſame Quantity are equal one to another.*

Suppoſe $A=Z$ and $B=Z$; that is A equal to Z and B alſo equal to Z, then A and B are two equal Quantities, this reaſoning is expreſſed thus: if $A=Z$ and $B=Z$; or (which is the ſame) if $A=Z=B$: then $A=B$. I ſhall often uſe this Expreſſion: it muſt therefore be well obſerved. *To this* AXIOM *may be added the following one, which is no leſs Evident*: If A is equal to B, every quantity greater or leſs than B, is likewiſe greater or leſs than A.

4. IF *equal Quantities are added to equal Ones, their Sums will be Equal.*

If $A=B$, the ſame Quantity X being added to A and B, they will continue equal $A+X=B+X$.

5. IF *from equal Quantities equal ones be Subtracted, the Remainders are equal.*

If $A=B$, then $A-X=B-X$; that is if A and B are two equal Quantities, then A leſs X is equal to B leſs X.

6. IF *equal Quantities be added to unequal Ones, they will continue unequal, greateſt that which was greater, and leſſer that which was leſſer.*

If X and Z are two unequal Quantities, and A and B two equal Ones, $X+A$ and $Z+B$ are unequal, that greater or leſſer than the Other, according as they were before.

7. IF *equal Quantities be ſubtracted from unequal ones, the Remainders will be unequal.*

That is, if X and Z are unequal Quantities, $X-A$ and $Z-A$ are unequal, one greater or leſſer than the other, according as X and Z were before.

8. *A Quantity having the Sign* +, *being added to itſelf or its Equal having the Sign* —, *is equal to nothing, or Cypher, That is,* $+A-A=0$.

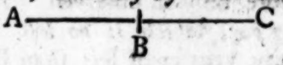
9. THINGS *which are the half or third &c. of the ſame or equal Quantities, are Equal, unequal if the whole Quantities, are unequal, and Greater or Leſſer, according as the Wholes are greater or leſſer.*

10. QUANTITIES *which every way coincide together when placed one upon the other, are equal.*

If two Lines placed one upon the other do exactly coincide, they are equal.

ADVERTISEMENT.

To *theſe Axioms may be added this Propoſition, that a thing is true by Conſtruction, when it is done exactly according to the Rule which was agreed upon, Thus if it be agreed upon to divide a Line into two equal Parts and that A C be ſo divided at the point B, then by Conſtruction A B and B C are each, the half of that Line A C.*



A Propoſition is alſo inconteſtable when it cannot be denyed without advancing an Abſurdity, or what is manifeſtly falſe.

ELEMENTS

ELEMENTS

OF

GEOMETRY,

BOOK I. SECTION I.

OF THE

Different Species of EXTENSION,

DEFINITIONS.

I. THERE *are three Sorts of Extenſion,* Length, Breadth, and Depth, *which are the three Dimenſions of Body.*

The Object of *Geometry* is not that material Extenſion of ſuch Bodies as are really extended into *Length, Breadth,* and *Depth,* it is ſuch an intell~~igible~~ectual Extenſion, as the Mind conceives, ſo that were there no Bodies in the World, yet what Geometers demonſtrate of Extenſion would nevertheleſs be true; hence it is, that altho' Bodies are liable to alterations yet the Truths of Geometry remain immutable; for they do not depend upon Matter, but upon clear and diſtinct Ideas which are in the Mind, therefore altho' there is no Body but what hath three Dimenſions, yet one may be conſidered without the other, *Length* may be conſidered without *Breadth,* and *Breadth* without *Depth,* as the length of a Road may be conſidered without reflecting on its Breadth, and the Breadth without thinking on the Depth of the Earth; the Idea of Length excludes thoſe of Breadth and Depth, and that of Breadth excludes that of Depth, and theſe Notions are true, altho' in reality thoſe three things are inſeparable, becauſe in the Manner they are conceived, they are diſtinguiſhed in this, that the one is conſidered without the other, therefore Geometricians can in this manner ſuppoſe things to be Long without being Broad, and Broad without being Deep, or Thick. Should any one aſſert that theſe Suppoſitions are utterly falſe, yet the Conſequences which are deduced from them, are not to be rejected as Erroneous. *For Example,* Altho' there is not in fact an exact Circle in the World, yet according to the Suppoſition of a Circle being a Figure whoſe *Circumference* is every where equally diſtant from its *Center,* it muſt needs follow, that all lines drawn from its *Center* to its *Circumference* are equal.

2 II. *A Point is that which hath no part.*

That is, the parts thereof are not conſidered; for every thing which is extended hath parts: Now if the Point be ſomething, it is extended, therefore it is ſaid to be without parts becauſe what it hath are not regarded, it being conſidered as indiviſible.

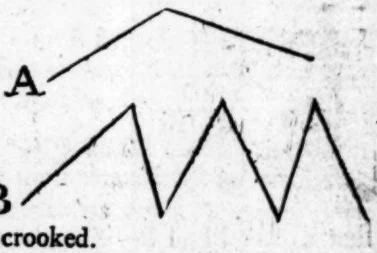
3 III. *A Line is Length without Breadth.*

That is, an Extenſion which is ſuppoſed either to have no breadth or whoſe breadth is not conſidered, it may alſo be conceived to be the Trace or Fluxion made by the Motion or Change of Place of a Point. *There are two ſorts of Lines, the right Line, and curve Line.*

4 IV. THE *right Line is the Shorteſt which can be drawn between two given points.*

5 V. CURVE LINES *are thoſe which are not the Shorteſt between two given points.*

There are many Sorts of curve Lines, regular and irregular, Geometrical and Mechanical, but theſe we ſhall not treat of here. Lines which are compounded of two or more Lines as A and B are called crooked, the Line A compoſed of two Lines, and the Line B made of ſeveral Lines, may either of them be conſidered as only one Line, therefore to diſtinguiſh them they muſt be called crooked.



6 VI. *A Surface is an Extenſion long and broad without Depth.*

7 VII. *A right or plane Surface, is the Shorteſt between two right Lines.*

A right Surface may be conceived to be generated by the Motion of a right Line.

8 VIII. *A Curve Surface is greater than the right or plane Surface; which is between the ſame two Lines.*

9 IX. *A Solid is an Extenſion which hath three Dimenſions, Length, Breadth, and Depth.*

A Solid is generated by the flux or motion of a Surface. Every Solid is a Body, as the very Name imports; for that is Extenſion which hath Length, Breadth and Depth, therefore, Extenſion and Body is the ſame Thing.

SECT.

SECT. II.

Of Length which is the firſt and ſimpleſt Dimenſion of Body.

Of right Lines.

Evident *Propoſitions concerning right Lines, or Corallaries proceeding from the Definition thereof.*

Propositions.

I. THE *Ends or Extremities of a Line are two Points.* 10

II. When *two different Lines interſect, their Section is an indiviſible Point.* 11

III. If *a Line drawn between two Points, is farther from one Part than another of a right Line drawn between the ſame two Points, it is longer than that right Line.* Plate 1. Fig. 1. 12

The Curve Lines A C B and A D B are longer than AB, and the crooked Lines E F G and E H G are longer than E G, this follows from the true Idea of a right Line, which is the ſhorteſt that can be drawn between the ſame two Points.

IV. Two *Points being given, to draw a right Line from one to the other.* 13

This is eaſily done by a Rule, or a Thread ſtretched from one Point to the other.

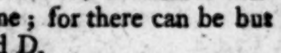
V. *A right Line given may be prolonged.* 14

This may alſo be done by help of a Rule. But when the given Line is very ſhort, it will be difficult to place the Rule ſo that the produced Part may be in a direct Line with the given one, I ſhall therefore ſhew how to do it geometrically, when we come to the *Recreative Problems* at the End of *Book* V.

VI. Between *the ſame two Points can be drawn but one right Line.* 15

VII. Two *right Lines having two Points common to each other, are only one and the ſame right Line.* 16

B C and A D have two Points common, as A and B, between which it is impoſſible to conceive any more than one right Line; therefore AB and BA make but one Line. BA being prolonged cannot extend itſelf elſewhere than towards C, nor AB but towards D, wherefore AD together with BD makes only one right Line; for there can be but one right Line between the two Points C and D.



VIII. Therefore *the Poſition of a right Line depends only upon two Points.* 17

For if a Line be drawn through the two given Points, it will be the Line required, as appears evident, by the preceeding Propoſition.

18 IX. Two *right Lines which croſs, or interſect each other, can meet only at the Point where they interſect.*

19 X. The *Part of a right Line, is a right Line.*

When the Part of a right Line is mentioned, it implies ſomewhat more than a Point.

SECT. III.

Of the Circular Line.

ADVERTISEMENT.

THERE *being innumerable Sorts of Curve Lines, I ſhall here conſider only the Circular Curve, which next to the right Line, is the ſimpleſt and eaſieſt to be underſtood.*

DEFINITIONS.

20 I. *A Circle is a Line having neither Beginning nor End, and which is in all its Parts equally diſtant from a Point within the ſame, which Point is called the Center thereof.*

21 II. The *Circle conſidered as a Surface, is all that Space which is contained within the Line above deſcribed.*

22 III. The *Line which bounds the Surface of the Circle, or the Circle conſidered as a Line, is called the Circumference.*

23 IV. All *Lines which croſs the Surface and paſs through the Center, are called Diameters.*

24 V. Every *Line which iſſues from the Center and terminates at the Circumference, is called a Radius, or Semi-diameter.*

25 VI. Lines *that terminate at the Circumference and which do not paſs through the Center, are called Chords.*

26 VII. That *Part of the Circumference which is cut off by a Chord Line, is called an Arc.*

When a Chord Line paſſes through a Circle, it divides it into two unequal Portions, which are both bounded by the Extremities of that Chord Line. When the Chord of an Arc is mentioned, it ſignifies the leſſer Arc, unleſs the other be particularly expreſſed.

A the Center. B the Circumference. DAC a Diameter.
AC a Radius, or Semi-diameter. EF a Chord Line. *Fig.* 2.

27 VIII. The *Circumference of every Circle is ſuppoſed to be divided into three hundred and ſixty equal Parts called Degrees, each Degree is divided into ſixty leſſer Parts called Primes, or Minutes, each Prime, or Minute, into ſixty Seconds, and each Second, into ſixty Thirds, and ſo on, ad inf.*

28 IX Circles *which are deſcribed from the ſame Center, are concentric to each other, and thoſe which have not the ſame Center, are called Excentrics.* Fig. 3.

Z and X

Z and X being deſcribed from the ſame Center A, are Concentrics, but E and F, whoſe Centers are two different Points B and D, are Excentrics.

EVIDENT Propoſitions concerning the Circular Line.

29 I. THE *Circumference of a Circle may be deſcribed with any given Interval, or Diſtance.* Eucl. 1. Prop. 2 and 3. and Book 4. Prop. 1.

THE Inſtrument commonly uſed for deſcribing a Circle, is a Pair of Compaſſes; with which, it is evident, a Line may be taken equal to a Line given; and of two unequal Lines from the greater may be taken a Line equal to the leſſer, which make the three Propoſitions in Euclid, as quoted above.

30 II. IN *the ſame or equal Circles, the Chords of equal Arcs are equal, and equal Chords correſpond to equal Arcs.* Eucl. 3. Prop. 24.

THIS muſt needs be evidently ſo, for all the Parts of a Circle being compoſed in the ſame regular and uniform Manner, there cannot be any Difference conceived between them.

31 III. IN *the ſame, or equal Circles, the longeſt Chords ſubtend the greateſt Arcs, and the greateſt Arcs are ſubtended by the longeſt Chords.* Eucl. 3. Prop. 28 and 29.

32 IV. ARCS *containing an equal Number of Degrees are greater or leſſer, according as their correſponding Circles are greater or leſſer.*

THIS is evident, for the greater the Whole is, the greater ought its Part to be: the third Part of a Yard is greater than the third Part of a Foot.

33 V. THE *Chords of Arcs containing equal Degrees are proportionally longer or ſhorter, according as their correſponding Circles are greater or leſſer.*

34 VI. THE *Diameter of a Circle divides it into two equal Parts, each of which is called its Semi-circumference.*

35 VII. ALL *the Radii, Semi-diameters, or Lines drawn from the Center of a Circle to its Circumference being equal, it from hence follows, that all Lines drawn from the Center which are ſhorter than the Radius, will have their Extremities within the Circle, and ſuch as are longer than the Radius will proceed beyond the Circumference, but ſuch as are of the ſame Length, will juſt reach thereto.*

36 VIII. EQUAL *Circles have equal Radii.*

FOR Circles are greater or leſſer according to the Length of their Radii.

37 IX. TWO *Circles deſcribed with the ſame Radius and from the ſame Center, are the ſame, and no Ways diſtinct one from the other.*

'TIS the ſame here as of two right Lines, which make only one when drawn between the ſame two Points.

SECT. IV.

OF the different Poſition of two right Lines with reſpect one to another.

ADVERTISEMENT.

THERE are but three different Poſitions of two right Lines. 1. Either they may meet together, 2. Or not meet, 3. Or they may croſs each other, when they meet together, they may either incline, or not incline towards each other. All which Poſitions are here conſidered.

OF PERPENDICULAR LINES.

38 DEFINITION. *A Line which ſtands upon or cuts another Line, ſo that it inclines not more to one Side thereof than to the other, is called a Perpendicular.*

EVIDENT Propoſitions concerning Perpendicular Lines.

39 PROP. I. *If the Top of a Line which ſtands upon the Middle of another Line, be at an equal Diſtance from the two Ends thereof, then it does not incline to either Side, but is Perpendicular thereto.* Fig. 4.

If A, the Top of the Line AE, which ſtands upon E, the Middle of BD, is equally diſtant from B and D, the Extremities of the Line BD, it is then Perpendicular thereupon: this follows from the Idea of a Perpendicular Line, as given by the preceeding Definition.

40 II. IF *there are two Points in a Line which are equally diſtant from the Extremities of the Line which it ſtands upon, every Point in that firſt Line, will be equally diſtant from the ſame Extremities of the ſecond Line.*

IF the two Points A and E of the Line AE, are equally diſtant from B and D, all the other Points of AE are equally diſtant from B and D, for there is no Point in the Line AE, which can be conceived nearer B than to D, without imagining AE to be curved in that Point, and therefore not a right Line, which it was ſuppoſed to be.

41 III. IF *one of the Points of a Perpendicular Line, be equally diſtant from two other Points of the Line which it ſtands upon, all its other Points are equally diſtant from thoſe ſame two Points.*

If AE is Perpendicular upon BD, and one of its Points A or E be equally diſtant from B and D, the other will be alſo equally diſtant from the ſame Points B and D, for if A be equally diſtant from B and D, and E is not, then AE will incline more to one Side than to the other; and therefore is not Perpendicular, whereas the Suppoſition required it to be ſo.

42 IV. TO *demonſtrate that one Line is Perpendicular to another, it is ſufficient to ſhew, that two Points of the one, are each at an equal Diſtance from two Points of the other.*

To demonſtrate that AE is Perpendicular upon BD, it is ſufficient to prove, that its Points A and E are each at an equal Diſtance from B and D.

43 V. If *a Line which is Perpendicular upon another Line be prolonged, its produced Part will alſo be Perpendicular upon the ſaid Line.*

AE is Perpendicular upon BD; its produced Part EC is alſo Perpendicular upon BD, for it is only one and the ſame right Line: it cannot poſſibly be conceived to be otherwiſe, by reaſon AC does not bend either towards B or D.

44 VI. If *one Line is Perpendicular upon another, then are thoſe Lines reciprocally Perpendicular to each other.*

If AC is Perpendicular upon BD, the Line BD is Perpendicular upon AC, for B cannot be conceived to incline more towards A than to C, without imagining D at the ſame Time inclined moſt to C; and were it ſo, than A would incline more towards B than to D, therefore AE will not be Perpendicular upon BD, which is contrary to the Suppoſition, therefore BD is Perpendicular upon AC, as is alſo AC upon BD.

Problem. I.

45 To *draw a Line Perpendicular to a given Line, from a given Point without the ſame.* Eucl. 1. Prop. 12.

Let the given Point be K, from which let there be drawn a Line Perpendicular upon Z. 1. From the given Point K as a Center, draw the Arc BC croſſing the given Line in the Points B and C, which being both in the Line Z, and in the Circumference of a Circle whoſe Center is K, they are therefore equally diſtant from K. 2. From C as a Center, with the Diſtance CK deſcribe an Arch, and from B with the ſame Extent, croſs that ſame Arch whoſe Points of Interſection are K and D, which are equally diſtant from B and C. 3. Draw a right Line from K to D, then two of the Points of that Line K and D, being *by Conſtruction*, equally diſtant from B and C, it muſt be, as has been proved, * that the Line KD is Perpendicular upon Z; *which was to be done.*

Problem. II.

46 To *erect, or raiſe a Perpendicular from a Point given in a Line.* Eucl. 1: Prop. 11.

From the Point K in the Line Z, let there be raiſed a Perpendicular. 1. From K as a Center, draw a Circle ſo as to croſs the Line Z in two Points, as at A and B. 2. From theſe two Points A and B, deſcribe two other Circles with any equal Extent taken at Pleaſure, ſo that the ſaid Circles may interſect each other, as here at D. 3. Draw a Line from the Point D to K, which is the Perpendicular required: For by Conſtruction, D is equally diſtant from A and B, whoſe Point K is alſo by Conſtruction, equally diſtant therefrom, therefore the Line DK, having two of its Points equally diſtant from A and B, it is Perpendicular thereupon.†

---

* *Sup. n.* 43. † *Sup. n.* 43.

When the given Point is at or near the extremity of the given Line, as A, take any Point at pleaſure as C, and open the Compaſſes to the diſtance of A C, and from C as a Center deſcribe a Circle, mark the Point where that Circle croſſes the given Line, as here at B, and draw through B and C the Diameter B D, then will a Line drawn from D to A be the Perpendicular required.

CORALLARY.

47 HENCE *follows the method of dividing a right Line into two equal Parts.* Eucl. 1. Prop. 10.

SUPPOSE the given Line be A B. from A and B as Centers, and with the ſame Extent taken at pleaſure deſcribe two Circles which interſect here at C and D, then by reaſon that C and D are equally diſtant from A and B, a Line drawn from C to D will be Perpendicular upon A B.* Now the Point E common to the two Lines D C and A B, is equally diſtant from A and B, † therefore A E is equal to E B; conſequently the Line A B is divided into two equal parts.

48 THEOREM. 1. THERE *cannot be more than one Perpendicular raiſed upon a Line from the ſame Point.*

A D is Perpendicular upon the Line B C. It is to be demonſtrated, that there can be no other Perpendicular raiſed from the Point A, as for Example, A E or any other whatſoever,

From A as a Center, deſcribe the Circle B F D E G C. The Points B and C are equally diſtant from A, and the Point D, where the Circle cuts the Perpendicular, is equally diſtant from B and C ‡. Therefore D B=D C, the two Arcs B F D and C G D are alſo equal,‖ Conſequently D is in the middle of the Arc B F D E G C. If E A is Perpendicular to B C, then by the ſame Reaſon, BE=CE, and B F D E=C G E; conſequently E will be alſo in the middle of B F D E G C, therefore B F D=B F D E, *which is abſurd.*

49 THEOREM 2. *A Line let fall Perpendicularly upon the Middle of another Line, paſſeth thro' all the Points which are equally diſtant from the Extremities of that other Line.*

THE Line AD is let fall Perpendicularly upon A the middle of B C. It muſt be proved, that it paſſeth thro' all the Points which are equally diſtant from B and C, the extremities of B C. If you deny it, and would aſſert that the Point E, which A D doth not paſs thro', is equally diſtant from B and C, then let there be drawn a right Line from that Point E to A, which will be Perpendicular upon B C, ſeeing that two of its Points A and E are equally diſtant from B and C,⸸ Now it was demonſtrated by the preceeding Theorem, that there cannot be drawn two Perpendiculars from the Point A, therefore it is not true, that the Point E is equally diſtant from B, and C.

---

* *Sup. n.* 43. † *Sup. n.* 41. ‡ *Sup. n.* 42.
‖ *Sup. n.* 31. ⸸ *Sup. n.* 43.

THEOREM. 3. FROM *the ſame Point to the ſame Line, there can be drawn but one Perpendicular.* 50

THE Line A D is Perpendicular upon the middle of B C, I ſay, that from the Point D, no other Line can be drawn Perpendicular upon B C, for every other Line drawn thereto from D, will fall either on one ſide or the other of the Point A, ſuppoſe at E, then if E A is a Perpendicular, the Point E muſt be equally diſtant from B and C, ‖ therefore B E is the half of the Line B C, but A B is alſo the half, therefore B A=B E, *which is abſurd* therefore, &c.

THEOREM 4. TWO *Lines which are Perpendicular upon a third, can never meet.* 51

FOR if they cou'd meet or interſect, there would be at that Point of meeting or interſection, two Perpendiculars upon the ſame Line, *which has been demonſtrated to be impoſſible.*

ADVERTISEMENT.

THE *Diſtance between a Point and a Line, is meaſured by a Perpendicular, becauſe it is the ſimpleſt and moſt certain Meaſure, by reaſon there can be but one Perpendicular drawn from a Point to a Line, and beſides, it is the ſhorteſt of all lines which can be drawn from the ſame Point to the ſame Line, as will be made appear by the following.*

THEOREM 5. THE *Perpendicular is the ſhorteſt of all Lines which can be drawn from the ſame Point to the ſame Line.* 52

THE Line B A is Perpendicular upon Z, It is to be demonſtrated, that it is ſhorter than any other drawn from the Point B to the Line Z. Prolong B A unto C, making B A=AC, the Line D A is Perpendicular upon B C, as is B C likewiſe upon A D * The Point D is therefore equally diſtant from B and C,† therefore B D is equal to D C; But the right Line B C is ſhorter than the Line B D+DC‡ conſequently A B, the half of B C, is ſhorter than B D, the half of B D+D C, Q. E. D.

OF OBLIQUE LINES.

DEF. 1. *A Line which meets another Line, ſo as to incline more to one ſide thereof than to the other, is called an Oblique Line.*

DEF. 2. *B C is an oblique Line upon Z; having drawn from its extremity B, the Perpendicular A B, the Line A C which is between A, the Foot of the Perpendicular, and C that of the Oblique, is the diſtance between the oblique Line and its Perpendicular, and that diſtance is the meaſure of the Obliquity of B C. Therefore a Line is more or leſs oblique, in proportion to the diſtance thereof from its Perpendicular.*

LEMMA. 1. TWO *right Lines having upon each of them a Perpendicular, if they be placed one upon the other, ſo as the foot of one Perpendicular be directly upon that of the other, the two Perpendiculars will concide together.*

‖ *Sup. n.* 42. * *Sup. n.* 42. † *Sup. n.* 45. ‡ *Sup. n.* 12.

AB is Perpendicular upon Z, and ab upon X, *'tis to be proved*. That if X be laid upon Z So that a be put upon A, the two Perpendiculars AB and ab will coincide, for ſince that after the above mentioned Poſition, X and Z become only one Line, AB will be Perpendicular upon each of them, ſo alſo will ab, therefore if AB and ab did not coincide, there would be two Perpendiculars upon the ſame Line at the ſame Point, *which is impoſſible.**

57 Lemma. 2. If *from the Ends of a Line be drawn four Lines, two of which unite in a Point nearer the given Line than the other two, I ſay, that the two laſt taken together will be longer than the two firſt.*

Let BC be the Line given, and from the Points B and C, draw the two Lines BD and CD, and the two others BE, CE, I ſay, that $BE+CE$ is longer than $BD+CD$. A right Line being the ſhorteſt between two points. † $CE+EG$ is longer than $CD+DG$, and by the ſame Reaſon $BG+GD$ is longer than BD, therefore $CE+EG+GD+BG$ is longer than $CD+DG+BD$, taking DG away from each, the remainder $CE+FG+BG$ or $CE+EB$, will, *according to the ſeventh Axiom*, be longer than $BD+DC$, *which was to be proved.*

58 Theorem I. When *the Perpendiculars and their Obliquities are equal, the Oblique Lines are equal.*

AB is Perpendicular upon AC, and ab upon ac, the two perpendiculars are equal, alſo AC the diſtance of the Perpendicular AB, is equal to ac, the diſtance of the Perpendicular ab. *it is to be proved*, that $BC=bc$. Having according to *Lemma* 1. Placed the two equal Lines ac, and AC upon each other, the Perpendicular ab will coincide with the Perpendicular AB, and c with C, ſo likewiſe bc with BC, Q. E. D.

59 Theorem. 2 Oblique *Lines drawn from the ſame Point to the ſame Line, are in length Proportional to their diſtance from the Perpendicular.*

It is to be proved, That BE is longer than BD, to do which, let BA be prolonged even to C ſo that $AB=AC$, then ‡ $BD=DC$, and $BE=EC$, now $BE+EC$ is longer than $BD+DC$ *by Lemma* 2; therefore BE the half of $BE+EC$, is longer than BD the half of $BD+DC$, *according to Axiom* 9.

Theorem. 3. An *Oblique Line is longer, according to the Length of its Perpendicular and its Diſtance therefrom; and if its Diſtance from the Perpendicular be the ſame, the longer the Oblique Line is, the longer is its Perpendicular.*

1. Let BC and DE be two Oblique Lines, the Perpendicular AB is longer than AD, and AC its Diſtance from the Perpendicular BA, is longer than AE that of DA, I ſay, that the Oblique Line BC is longer than the Oblique DE, *for (by the preceeding Theorem,)* BE is longer than BA, and ſince that AC is Perpendicular upon AB, *by the ſame Theorem*, BE is longer than DE, conſequently, BC being longer than BE, is alſo longer than DE. 2. AE is the ſame Diſtance, I ſay, that if BE is longer than DE, the Perpendicular AB is longer

* *Sup. n* 49. † *Sup. n.* 12. ‡ *Sup. n.* 58.

is longer than the Perpendicular AD, for if AD were equal or longer than AB, then, either by the firſt or precedent Theorem, BE would be ſhorter than DE, or elſe equal thereto, contrary to the Suppoſition which required BE to be longeſt.

THEOREM 4. *If the Perpendiculars and Oblique Lines are equal, their Diſtances from the Perpendiculars are equal.* 61

LET AB = ab, and BC = bc, *It muſt be proved,* that AC = ac, Having placed ab upon AB, the two equal Lines will entirely coincide, and *by Lemma* 1, the Perpendicular ac will by this Means partly coincide with AC. If AC is ſhorter than ac, and that c does therefore not coincide with C, but with D, then bc will coincide with BD, but BD > BC, * which is contrary to the Suppoſition BC = bc, 'twou'd have been an equal Abſurdity to have ſuppoſed AC longer than ac, therefore it muſt be, that AC = ac, Q. E. D.

THEOREM 5. *If the Oblique Lines and their Diſtances from the Perpendiculars are equal, the Perpendiculars are equal.* 62

LET AC = ac, and BC = bc, *It is to be proved,* that AB = ab, Let AC be placed upon ac, the two equal Lines will coincide, and the Perpendicular ab with the Perpendicular AB in part at leaſt, *by Lemma the firſt,* † If it is ſaid that AB < ab, and that therefore b coincides with D, then bc will coincide with DC and is therefore equal thereto, and therefore DC ‡ longer than BC, which was ſuppoſed equal, the Concluſion would have been equally abſurd, if AB had been ſuppoſed longer than ab, therefore ab muſt entirely coincide with AB, and therefore AB = ab, *which was to be proved.*

OF PARALLEL LINES.

DEFINITION. TWO *right Lines which are equally diſtant in all their Parts one from the other, are called Parallels.* 63

EVIDENT Propoſitions concerning Parallel Lines.

PROP. I. *A right Line which hath two of its Points equally diſtant from another right Line, is parallel thereto.* 64

FOR the Poſition of a right Line depends only on two of its Points, therefore the two Lines being equally diſtant from each other, they are parallel, *according to the Definition.*

PROP. II. PERPENDICULARS *which are between two Parallels are equal.* 65

THE Perpendiculars meaſure the Diſtance between the two Parallels, which being every where the ſame, they are equal.

PROP. III. TWO *parallel Lines will not meet, ſuppoſing them prolonged,* ad inf. 66

FOR they cannot meet unleſs they approach at one end, they are therefore then no longer at the ſame Diſtance, conſequently ceaſe to be (*as ſuppoſed*) parallel.

PROP. IV. TWO *right Lines which are not parallel, but nearer at one end than the other, if ſufficiently prolonged will meet.* 67

ADVER-

---

* *Sup. n.* 59. † *Sup. n.* 56. ‡ *Sup. n.* 59.

ADVERTISEMENT.

THIS *is to be underſtood only of right Lines, for there are Curve Lines of ſuch a Nature, that they continually approach toward right Lines without ever meeting them, (as is demonſtrable) and theſe right Lines are called the* Aſymptotes *of the Curves.*

68 PROP. V. TWO *Lines which are Perpendicular upon one and the ſame Line, are parallel one to another.*

FOR two Perpendiculars can never meet.* Now if they were not parallel, they would meet according to the *preceeding Propoſition*, therefore they muſt be parallel.

69 LEMMA 1. LINES *which are between two parallel Lines, if they are Perpendicular to one parallel, they are ſo likewiſe to the other.*

IF AB Perpendicular upon X is not ſo upon Z, then Z is not ſo upon AB;* therefore being inclined to the Line AB, it will approach either towards one end or the other of the Line X, and meet it,† conſequently it is not parallel thereto, contrary to the Suppoſition which required it to be ſo.

70 LEMMA 2. THE *Line AB cannot be Perpendicular upon both Z and X, unleſs the two Lines are parallel.* §

FOR the two Lines (*ſame Figure*) are reciprocally Perpendicular upon AB;‡ and therefore they are parallel.

71 LEMMA 3. IF *between two right Lines there are two other equal right Lines, of which one is Perpendicular upon the firſt, and the other upon the ſecond, I ſay, that the two firſt Lines are parallel.*

BETWEEN Z and X, are two equal Lines AB and CD, one of which, namely AB is Perpendicular upon X, and CD upon D; I ſay, that Z and X are parallel. For if Z is not parallel to X, it either declines from it, or inclines to it, both which ſhall be ſhewn to be abſurd, therefore it muſt be that they are parallel. 1. If it be ſuppoſed to decline therefrom, From the Point A having drawn AE Perpendicular upon Z,* then if this Perpendicular coincide with AB, Z and X are Perpendicular upon AB *by Conſtruction*, and therefore parallel to each other † contrary to the *Suppoſition*, That if AE fall on one ſide of AB, the Perpendicular AE will be ſhorter than the Oblique Line AB,‡ conſequently ſhorter alſo than CD = AB, § therefore the Line Z inclines to the Line X, contrary to the Suppoſition. 2. If it is ſuppoſed that the Line Z inclines to the Line X, 'twill be a like Abſurdity; for having raiſed upon the Line Z the Perpendicular BE at the Point B, ‖ as AB is Perpendicular upon X, if it coincides with AB, then according to what has been ſaid, Z and X are parallel, but if BE falls on one ſide, as AB was propoſed Perpendicular upon X, BE will be Oblique, and conſequently longer than BA, ¶ or than its equal CD, therefore Z and X decline from each other, contrary to the *Suppoſition* which made them incline, Q. E. D.

---

* *Sup. n.* 52.

* *Sup. n.* 45. † *Sup. n.* 67. ‡ *Sup. n.* 45. § *Sup. n.* 68.

* *Sup. n.* 46. † *Sup. n.* 68. ‡ *Sup. n.* 53. § *Sup. n.*

Axiom 3. ‖ *Sup. n.* 47. ¶ *Sup. n.* 51.

Problem. To *draw a Line parallel to a Line given through a given Point*. Eucl. 1. Prop. 31. 72

Let AD be the Line to which a Parallel muſt be drawn through the Point C, From the Point C let fall a Perpendicular upon the given Line AD, upon which having raiſed a Perpendicular ſuch as AB equal to CD, it is evident, that the Line which paſſeth through the two Points B and C will be parallel to A D, *according to the Definition*.[*]

Or *from any Point whatſoever as A, let ſuch an Arc be deſcribed as will paſs through the given Point C, from which with the ſame extent AC, having made the Arc AB equal to DC, the Line drawn through the Points B and C will be the parallel required.*

Theorem. Two *Lines parallel to a third, are parallel to each other*. Eucl. 1. Prop. 30. 73

X and Y are parallel to Z. From X draw AB Perpendicular upon Z, which being prolonged to C, ſeeing that it is Perpendicular upon Z it will be ſo upon Y, (a parallel to Z;[†]) and ſince Z is a parallel to X, the Line which is Perpendicular upon Z will be ſo likewiſe upon X, (a parallel to Z.[‡]) therefore ſince X and Y are Perpendicular upon AC, they are parallel to each other.[§] Q. E. D.

Corallary. Two *different Lines which are parallel to a third, cannot be drawn through the ſame Point*. 74

For by the *Theorem* they muſt be parallel to each other, which ſince they would have one Point common is abſurd, and it is the Property of Parallels never to meet.

SECT. V.

Of the different Poſition of two Circles with reſpect to each other.

ADVERTISEMENT.

One Circle may be ſo placed with reſpect to another Circle, 1. That they may neither interſect nor touch. 2. That they may interſect or not interſect. 3. That they may touch either on the inſide or outſide.

Evident Propoſitions concerning the Poſition of Circles.

Prop. 1. *Concentric-Circles do neither interſect nor touch.* 75

The Concentric Circles X and Z whoſe Center is A, may be conceived as deſcribed by the Points D and E of the Line AE, therefore the Circle which D deſcribes, is always within the Cir-

---

[*] *Sup. n.* 63. [†] *Sup. n.* 69. [‡] *Sup. n.* 69. [§] *Sup. n.* 68.

cle Z, which is deſcribed by E, the two Circles therefore cannot meet.

76 PROP. 2. TWO *Concentric-Circles are always equal diſtant.*

FOR between X and Z there is always the ſame diſtance DE. (ſame fig.)

77 THEOREM 1. TWO *Circles which interſect are not Concentric.* Eucl. 3. Prop. 5.

LET there be two Circles which interſect at the Points B and E, I ſay, that they have not the ſame Point for their Center. For if A be the Center of the two Circles which interſect at the Point B, the Lines AB and AC (*Radii of the ſame Circle*) are equal,* $AC = AB$. and by the ſame reaſon $AB = AD$. therefore $AC = AB = AD$; then $AC = AD$,† *that is,* the Part equal to the Whole, *which cannot be.*‡

78 THEOREM 2. TWO *Circles which touch are not Concentric, or have not the ſame Center.* Eucl. 3. Prop. 6.

LET there be two Circles which touch at the Point B, I ſay, they have different Centers. For if A was the Center of both the Circles, then $AD = AB$, and $AB = AC$,§ conſequently $AD = AB = AC$, therefore $AD = AC$,‖ *that is,* the Part AC is equal to its Whole AD, *which is abſurd.*

79 THEOREM 3. IF *two or more Circles have their Center in the ſame right Line, each croſſing it at the ſame Point, they can touch each other only at that Point.*

THE Circles X, Z, Y, have their Centers in the ſame Line which they croſs at the Point D. *it is to be proved,* that they can touch only at the Point D. If we ſhould ſay that they may touch elſewhere, or meet at the Point E, then $AE = AD$ ſeeing they are the Radii of the ſame Circle, by the ſame reaſon $CD = CE$, therefore $AE + CE = AD + CD$, *which is abſurd.** In like manner it is demonſtrated that Z and X do not meet in E, for if they ſo met, $BD = BE$, and $AE = AD$, but $AD = AB + BD$, or $AB + BE$, then $AE = AB + BE$, *which is abſurd.*†

80 CORALLARY. TWO *Circles cannot touch either within ſide, or without, but only in one Point.* Eucl. 3. Prop. 13.

X and Z touch within ſide, (*ſame Figure*) if it be at the Point D, they cannot touch elſewhere, for Example at the Point E, if they touch at E they can't touch at D, for $AB + BE$ will be equal to AE, *which is abſurd.*‡ X and Y touch on the out-ſide, if it be at the Point D, they cannot touch in any other, for Example in E, for then $AC = AE + EC$, *which is abſurd.*‖

81 THEO. 4. IF *two Circles touch on the inſide, the right Line which being produced will join their Centers, will paſs through their Point of Contact.* Eucl. 3. Prop. 11.

1. BY Theorem 2. Theſe two Circles have not the ſame Center. 2. Let F be the Center of BAB, if G which is not in the Line FA, which paſſeth through the Point of Contact, was the

---

* *Sup. n.* 20. † *Sup. Ax.* 3. ‡ *Sup. Ax.* 1. § *Sup. n.* 20. ‖ *Sup. Ax.* 3.

* *Sup. n.* 12. † *Sup. n.* 12. ‡ *Sup. n.* 12. ‖ *Sup. n.* 12.

Center of DAD; then $GA=GD$,[*] therefore $FG+GA=FG+GD$,[†] now $FG+GA>FA$ [‡] and conſequently longer than FB, for $FA=FB$,[‖] therefore $FG+GD>FB$ or $FG+GB$, then take from each the common Part FG, there will remain $GD>$ than the Whole GB,[§] *which is impoſſible.*

Theorem 5. If *two Circles touch on the out-ſide, a right Line drawn through their Centers, will paſs through their Point of Contact.* Eucl. 3. Prop. 12. 82

Let DAD and EAE be the two Circles which touch at A, I ſay, that the Line which will join their Centers will paſs thro' the Point A. If we deny this, we muſt unavoidably fall into an Abſurdity; for let B be the Center of DAD, and C that of EAE, and that therefore the Line BC will not paſs through A where the two Circles touch: $BA=BD$ and $CA=CE$,[*] then $BA+AC=BD+CE$, ſince that BC does not paſs through the Point of Contact of the two Circles which is common, D and E are not the ſame Point, there being between them the Diſtance DE, which add to $BD+CE$, then $BD+DE+CE$ is longer than $AB+AC$, *which is abſurd.*[†]

SECT. VI.

Of the Poſition of a right Line with reſpect to a Circle.

ADVERTISEMENT.

A Right Line may be wholly within, or without a Circle: If it is without, it may be in ſuch Poſition, that when prolonged it may either croſs it, or only touch without entering into it.

Theorem 1. If *any two Points be taken in the Circumference of a Circle, the right Line drawn from one of thoſe Points to the other, will fall wholly within the Circle.* Eucl. 3. Prop. 2. 83

B and C are two Points in the Circumference of the Circle X, it muſt be proved, that the Line BC drawn between thoſe Points, is entirely within the Circle. From A the Center of X, draw a right Line to D the Middle of BC; ſince AB and AC, the Radii of X, are equal, and ſince D is the Middle of BC, the Line AD is Perpendicular,[‡] and therefore ſhorter than either AB or AC,[‖] therefore D is within the Circle.[§] Now every other oblique Line drawn from A to BC, will alſo be ſhorter than either AB or AC,[*] all the Line BC is therefore within the Circle.

---

[*] *Sup. n.* 20. [†] *Sup. Ax.* 4. [‡] *Sup. n.* 12. [‖] *Sup. n.* 20. [§] *Sup. Ax.* 7.

[*] *Sup. n.* 20. [†] *Sup. n.* 12. [‡] *Sup. n.* 36. [‖] *Sup. n.* 40. [§] *Sup. n.* 53.

[*] *Sup. n.* 59.

84 **THEOREM 2.** IF *a Chord is cut into two equal Parts by a Perpendicular Line, I ſay, that Line cuts the Arc of the Circle into two equal Parts, and paſſes through the Center.*

LET the Chord CD be cut into two equal Parts by the Perpendicular Line BF, at the Point E, I ſay, that this Line cuts the Arc CD equally in two at B, and paſſes through the Circle's Center. 1. Seeing that the Point E is equally diſtant from the Points C and D, and that BF is Perpendicular upon CD, the Point D which is in this Perpendicular, is alſo equally diſtant from the Points C and D, † and therefore the Chord BC is equal to the Chord BD, and conſequently the Arc BC is equal to the Arc BD, ‡ therefore the Arc CBD is cut equally in two, it is the ſame with the Arc CFD. 2. The Perpendicular BF paſſeth thro' all the Points which are equally diſtant from C and D, ‖ now the Center of the Circle is equally diſtant from theſe two Points C and D, therefore BC paſſeth through the Center.

85 **CORALLARY 1.** IT *is evident, that in order to divide an Arc into two equal Parts, a Perpendicular muſt be rais'd upon the Middle of its Chord Line.* Eucl. 3. Prop. 30.

86 **CORALLARY 2.** THE *two Arcs contained between two parallel Lines are equal.*

THE two Arcs MC and ND, contain'd between the two parallel Lines or Chords MN and CD are equal; for the Diameter BF being drawn Perpendicular upon CD, * it will alſo be ſo upon MN, † therefore $BM=BN$, and $BC=BD$, ‡ the Arcs of equal Chords are equal, ‖ therefore the Arc $BC-BM=BD-BN$; but the Arc BC leſs the Arc BM is equal to the Arc CM, and likewiſe $BD-BN=ND$; therefore $CM=ND$, Q. E. D. but if inſtead of the Chord MN, we had ſuppoſed at B, a Tangent parallel to CD, the Arc BC, might more readily have been demonſtrated equal to the Arc BD.

87 **PROBLEM 1.** THREE *Points being given, to find the Center of the Circle which paſſes through theſe Points.* Eucl. 3. Prop. 1.

LET the three given Points be A, B, C, join them by two right Lines AB, BC, which we will conſider as Chords of the Circle ſought, and on the Middle of each of thoſe two Lines draw two Perpendiculars, I ſay, that the Point K, where thoſe two Lines interſect, is the Center of the Circle required, *which is evident by the foregoing Theorem.*

IF *the three given Points had been in a right Line, the Problem would have been impoſſible, as is evident, for then the two Perpendiculars being parallel, could not interſect.*

88 **CORALLARY 1.** IF *two Circles have three Points common, as A, B, C, then the reſt are all likewiſe common.*

THESE two Circles having the ſame Center (*viz.*) K, and being deſcribed with the ſame Extent, they cannot be different from each other.*

CORALLARY 2.

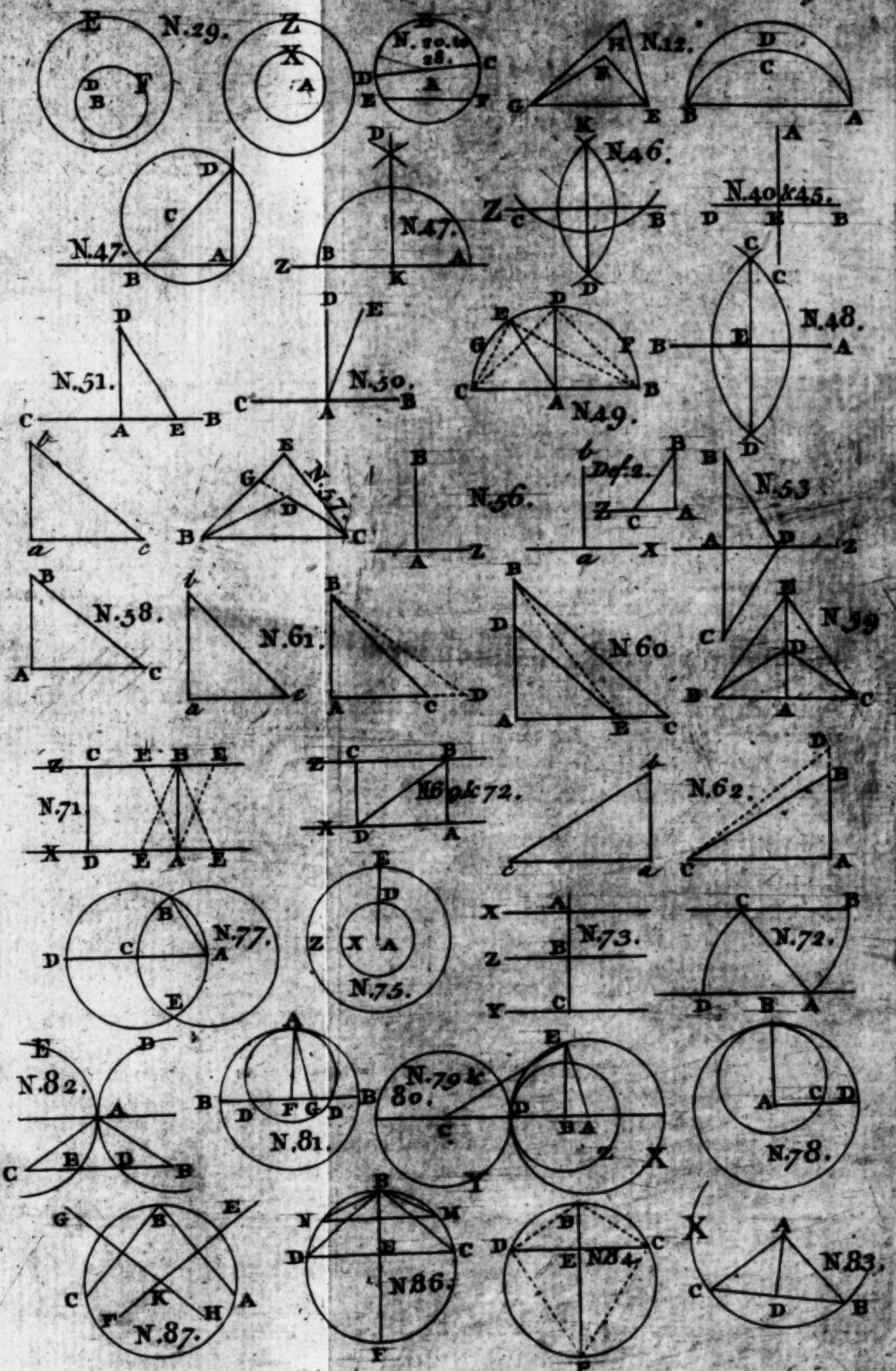
---

† *Sup. n.* 42. ‡ *Sup. n.* 31. ‖ *Sup. n.* 42.

* *Sup. n.* 46. † *Sup. n.* 69. ‡ *Sup. n.* 42. ‖ *Sup. n.* 31.

* *Sup. n.* 38.

Book 1. Plate 1.



CORALLARY. 2. TWO *Circles can interſect only in two Points.* 89
Eucl. 3. Prop. 10.

FOR if they cou'd interſect in three, they would have three Points common, therefore, by the preceeding Corallary, they would not be two different Circles.

CORALLARY. 3. PART *of a Circle being given, the whole Circle may be compleated.* Eucl. 3. Prop. 25. 90

MARK three Points in the Part given, then may the Center of the Circle whoſe part it is, be found by the preceeding Problem.

THEOREM. 3. IF *a Line cuts the Chord or Arc of a Circle into two equal Parts, and paſſeth thro' the Center, I ſay, that it cuts it Perpendicularly.* Eucl. 3. Prop. 3. 91

LET BK be the Line which cuts the Arc or Chord of a Circle into two equal Parts, and paſſes thro' the Center K, I ſay, that it cuts the Chord Perpendicularly, for in the Line BK there are two Points (*viz.*) A or B and K equally diſtant from C and D, ſince AD is half the Line BD, and BC half the Arc CBD, and that K is the Center, therefore BK is Perpendicular. *

THEOREM. 4. IF *a Line cuts the Chord of a Circle Perpendicularly, and paſſes thro' the Center, I ſay, that it divides it into two equal Parts.* Eucl. 3. Prop. 3. 92

THE Line BK paſſes thro' the Center K, and is Perpendicular upon CD, I ſay, that it cuts CD equally in two.

THE Center is equally diſtant from C and D, and BK being Perpendicular, the Point A, and all the other Points of BK, ought to be equally diſtant from C and D. § Then AC=AD, therefore CD is divided into two equal parts.

THEOREM. 5. TWO *Chords which don't paſs thro' the Center, cannot interſect ſo as to divide each other into two equal parts.* Eucl. 3. Prop. 4. 93

LET DE and BC be two Chords which interſect out of the Circle's Center at the Point A, I ſay, that they are unequally divided. If the two Chords interſect at A, which is not the Center, and that this Point be the Middle of the two Lines, then having drawn from A, a Line to K the Center, that Line KA, will be Perpendicular upon BC and upon DE. † Then BC and DE are Perpendicular upon KA, ‡ therefore, there is two Perpendiculars upon the ſame Point A, *which is impoſſible.* || 94

THEOREM. 6. CHORDS *which are equally diſtant from the Center are equal, and if they are equal, they are equally diſtant from the Center.* Eucl. 3. Prop. 14.

LET BC and GH be Chords equally diſtant from the Center K. I ſay, that they are equal, and if they are equal, their diſtances KE and KF from the Center are equal. 1. Upon theſe Chords I draw the Perpendiculars DK and KL, which interſect at the Middle. ⸸ *by the Hypotheſis,* KE=KF, and ſince that BK =KH, and KC=KG, then the Oblique Line BK being equal to the Oblique KH, and the Perpendiculars KE and KF belonging to thoſe Obliques being equal, BE and HF their diſtances from the Perpendiculars are alſo equal. * the ſame Way it is proved, that EC=FG, and that therefore BC=HG, *which was to be proved.* 2. It has been proved, that when Oblique Lines as KB and KH are equal, and that BE and HF their diſtances from the Perpendicular are alſo equal, that then the Perpendiculars KE and KF are equal. ¶

---

* *Sup. n.* 40. § *Sup. n.* 42. † *Sup. n.* 91.
‡ *Sup. n.* 45. || *Sup. n.* 49. ⸸ *Sup. n.* 92.

95 THEOREM. 7. THE *Diameter, or Line which paſſeth thro' the Center, is the longeſt of all Lines which are in the Circle.* Eucl. 3. Prop. 15.

LET the Line AB be the Diameter. It muſt be proved, that it is longer than CD, or any other Line whatſoever which doth not paſs thro' the Center, *which is plain.* For KC=KB, and KD=KA, therefore BA=KC+KD, now KC+KD is longer than CD, † therefore BA is longer than CD.

96 THEOREM. 8. CHORDS *neareſt to the Center of the Circle are longeſt, and the longeſt Chords are neareſt the Circle's Center.* Eucl. 3. Prop. 15

LET A be the Center, and the propoſed Lines be BC and DE, it muſt be proved, that DE which is neareſt the Center A, is longer than BC which is farther off. AM is the Diſtance of BC, and AN that of DE, the latter of which is Shorteſt, it muſt be proved, that DE which is neareſt the Center, is longer than BC which farther off, which is evident, for the Arc EFD is longer than the Arc CFB. Now the greateſt Arcs have the longeſt Chords, § therefore DE the Chord of EFD, is longer than BC the Chord of CFB, Q. E. D.

97 THEOREM. 9. IF *a Line is the common Chord of two Arcs of unequal Circles which interſect, I ſay, that the Arc of the leſſer Circle contains more Degrees than the Arc of the greater.*

LET X and Z be two unequal Circles which interſect at the Points C and D, their Chord CD is common; I ſay that the Arc CZD of the leſſer Circle, contains more Degrees than the Arc CXD of the Greater, for if the two Arcs of which CD is the Chord were the Same, each for Example of ten Degrees, then what has been before proved ‡ would be falſe, (*Viz.*) that the Chords of Arcs containing the ſame Number of Degrees, are proportionally Longer as their correſponding Circles are greater.

98 CORALLARY. THEREFORE *the ſame Line cannot be the Chord of two Arcs containing the ſame Number of Degrees, and which are portions of unequal Circles.*

IT was obſerved, *Sup. n.* 26. that the leſſer Arc of the Circle is to be underſtood, unleſs the other is directly mentioned.

99 THEOREM 10. IF *from a Point without the Circle, be drawn ſeveral Lines which croſs it, and end at the Circumference, I ſay,* 1. *Of all the Lines which fall upon the convex Part, that which being*

---

* *Sup. n.* 61. ¶ *Sup. n.* 62. † *Sup. n.* 12.
§ *Sup. n.* 32. ‡ *Sup. n.* 33.

being produced will paſs thro' the Center, is the Shorteſt, next, thoſe which are nearer that, are ſhorter than thoſe which are farther off. 2. It is juſt the Reverſe, of the Lines which fall upon the Concave Part. Eucl. 3. Prop. 8.

LET the Point be B, from which having drawn the Lines BH. BG. BF. I ſay, 1. That BC which paſſeth thro' the Center, is ſhorter than any other drawn from the ſame Point, than BD for Example. For $AC=AD$, now $AC+CB$ is ſhorter than $AD+DB$, * therefore BC is ſhorter than BD $AD=AE$. Now $AD+DB$ is ſhorter than $AE+EB$, ‡ then taking away the equal Quantities AD and AE, the Remainder DB is ſhorter than the Remainder BE. ‖.

2. THAT BH which paſſeth thro' the Center is longeſt, for $AB+AH=AB+AG$, now $AB+AG$ is longer than BG § now DG is longer than DF. † therefore BH is longer than BG, it is likewiſe evident, that $BD+DF$ is longer than BF, ‖ therefore $BD+DG$ is longer than BF.

CORALLARY. IF *there are two right Lines, equal or unequal,* 100
placed one upon the Other and which agree to a Point at one of their Ends, upon which Point if one of them be turned round, the Line which joins their other ends, will as that Line moves, continually encreaſe its length, even till the ſaid two Lines, make only one right Line.

LET AB, and AC be the two Lines upon AB one of which as AC, turn round upon the End A, in moving from AB it will paſs thro' D, E, F. The Lines which join their ends as BD, BE, BF encreaſe in length even 'till the ſaid two Lines form only the right Line BAH *which is what was to be proved.*

THEOREM. 11. IF *there be Lines drawn from a Point which* 101
is out of the Center of a Circle, to the Circumference, I ſay, that which paſſeth thro' the Center is the Longeſt, and the other Part remaining which will compleat the Diameter, is the ſhorteſt Eucl. 3. Prop. 7.

LET the Point be B, and let there be drawn thro' this Point the Lines BC, BF, BD, and BE, which paſſes thro' the Center A, I ſay, that BE is the longeſt, and its part BD the ſhorteſt of all thoſe which end in that Point. 1. $BA+AE=BA+AC$, ſeeing that $AE=AC$, now $AB+AC > BC$, * therefore BE equal to $BA+AC$ is longer than BC. 2. $AF=AB+BD$. now $AB+BF$ is longer than AF, ¶ then taking away the common part AB, the Remainder BF is longer than the Remainder BD. §

THEOREM. 12. THERE *is no Point in a Circle, except the Cen-* 102
ter, from which there can be drawn to the Circumference any more than two equal Lines. Eucl. 3, Prop. 9.

LET A be the Center of the Circle X, therefore the Point B is not the Center. 1. I ſay, that BD is longer than BC, † 2. That BE is longer than BD, for AD and AE are each the Radii of X, which in turning or removing from AB, ought to make

---

* *Sup. n.* 12. ‡ *Sup. n.* 57. ‖ *Sup. Ax.* 7.
§ *Sup. ax.* 7. † *Sup. n.* 96. ‖ *Y. ſ. n.* 12.
* *Sup. n.* 12. ¶ *Sup. n.* 12. § *Sup. Ax* 7. † *Sup. n.* 101.

BE longer than BD,† and in like Manner BF is longer than BE, now BG is longer than BF,* Conſequently, all Lines drawn from B to any part of the Semi-circumference of X, as CD, FG, &c. are unequal. Let us take the Arc CH equal to CD, ſince that AC is Radius, it will cut DH Perpendicularly,‡ and in two equal Parts at the Point I;§ therefore the Perpendicular and its Diſtance from the Oblique Line being equal, the Oblique Lines BH, BD are equal.‖ But every other Line except BH, will as has been proved, be either ſhorter or longer, in the ſame Manner it may be demonſtrated, that there can be drawn but two equal Lines from C.

103 CORALLARY 1. HENCE, *it is only from the Center of the Circle, that there can be drawn more than two equal Lines to the Circumference.*

104 CORALLARY 2. IN *every Circle, that Point from which three equal Lines may be drawn to the Circumference, is the Center of the Circle.* Eucl. 3. Prop. 9.

105 THEOREM 13. *A Line which croſſes two concentric Circles, whether it paſs through their Center or not, the Parts of the Line which are between the Circumferences of thoſe two Circles, are equal to each other.*

LET BC, BD croſs two concentric Circles, I ſay, that BM=NC and BF=ED. 1. For BC which paſſeth through the Center, this has been proved.* 2. From A draw a Perpendicular upon BD, then GD=GB, and GE=GF, † then GD—GE =GB—FG;‡ now GD—GE=DE, and GB—GF=FB, therefore DE=BF.

106 THEOREM 14. *A Line Perpendicular at the End of Radius touches the Circle, and that in only one Point.* Eucl. 3. Prop. 16 and 18.

BD is Perpendicular upon BK, it muſt be proved that this Line touches the Circle X only at the Point B. If you ſay that it touches it in a ſecond Point, as in C; draw a Line from K to C, which is not Perpendicular upon BD, ſince by reaſon there can be drawn but one Perpendicular from K upon BD, § it is therefore longer than the Radius BK, which is Perpendicular upon BD,‖ wherefore the Point C is out of the Circle X, therefore BD does not touch in the two Points B and C, but in one only, as B.

107 CORALLARY. *A Circle cannot be touched by any more than one right Line, at the ſame Point.*

Two different Lines cannot touch the Circle X at the ſame Point B; for if ſo, they by the Theorem, they would both be Perpendicular upon BK, *which is impoſſible.**

108 THEOREM 15. IF *a Line be drawn within a Circle perpendicularly upon the Point of Contact of the Tangent or touching Line, it paſſeth through the Center of the Circle.* Eucl. 3. Prop. 19.

---

† *Sup. n.* 100. * *Sup. n.* 101. ‡ *Sup. n.* 91. § *Sup. n.* 42. ‖ *Sup. n.* 58.

* *Sup. n.* 76. † *Sup. n.* 92. ‡ *Sup. Ax.* 5. § *Sup. n.* 51. ‖ *Sup. n.* 53.

* *Sup. n.* 49.

CD is a Tangent or touching Line, from C the Point of Contact, draw a Perpendicular within the Circle, I ſay, that it will paſs through the Center K. If you think it paſſes through B which is not the Center, I prove it falſe, having drawn from C the Radius KC, and raiſed at C a Perpendicular which will be a Tangent,† but it will be the ſame as CD, ſince there can be only one Tangent at the Point C, ‡ therefore at the ſame Point upon the Line CD, there would be two Perpendiculars CK and CB, *which is impoſſible.* ⸸

THEO. 16. *There cannot be drawn, a right Line between a Tangent and the Circumference of a Circle, but there may be drawn an infinite Number of circular Lines.* Eucl. 3. Prop. 16. 109

IF between the Tangent BD and the Circle X, there could be drawn a right Line, dividing the Space between the Tangent BD and the Circle, as the Line BF, upon that Line from the Point K, draw another Line Perpendicular thereto, *to wit* KE, which will be ſhorter than the Radius BK which is not Perpendicular upon that Line,* therefore KE being ſhorter than the Radius BK, its Extremity E is within the Circle, conſequently the Line BF cuts the Circle, and does not therefore divide the Space which is between it, and the Tangent BD.

BUT between the Tangent AB and the Circle Y, may be drawn an infinite Number of Circles, for having produced the Radius AC beyond the Center C, and from E as a Center, with the extent of EA having made the Circle Z, the Line AB will be Tangent thereto;† which Circle Z, being greater than Y, will be out of the Circle Y, in like Manner the Circle X whoſe Center is F, will be alſo between AB and Y, *&c. ad inf.* conſequently between the Tangent AB and the Circle Y, an infinite Number of circular Lines may be drawn.

CORALLARY. IT *is therefore evident, that the Space contained between the Tangent and the Circumference of a Circle, may be divided into an infinite Number of Parts.* 110

THEOREM 17. FROM *a Point which is without the Circle, there can be but one Tangent drawn to the Circle on the ſame ſide thereof.* 111

ALL other Lines will either interſect it or not reach thereto. Let A be a Point given out of the Circle, the Line AB touches it at the Point B, I ſay, that there can be no other Tangent drawn from A towards B, but that all other Lines will either interſect, or elſe not touch. 1. If a Line be drawn beyond AB, it will not touch the Circle, and 2. If it be drawn through the Point B, it is the ſame as AB. 3. And if it do not reach ſo far as the Point B, but goes through D, becauſe CB is longer than CD, then the Point D will be in the Circle, * therefore AD enters within the Circle, and cuts it.

---

† *Sup. n.* 106.

‡ *Sup. n.* 107.

⸸ *Sup n.* 31.

* *Sup. n.* 53.

† *Sup. n.* 106.

* *Sup. n.* 36.

112 Problem 2. To *draw a right Line ſo as to touch a Circle in a given Point.*

The Center is K, the given Point B, draw the Radius KB, and upon the End B,† raiſe the Perpendicular BD which will be the Tangent required.‡

113 Problem 3. To *draw a Tangent, from a Point given out of a Circle.* Eucl. 3. Prop. 17.

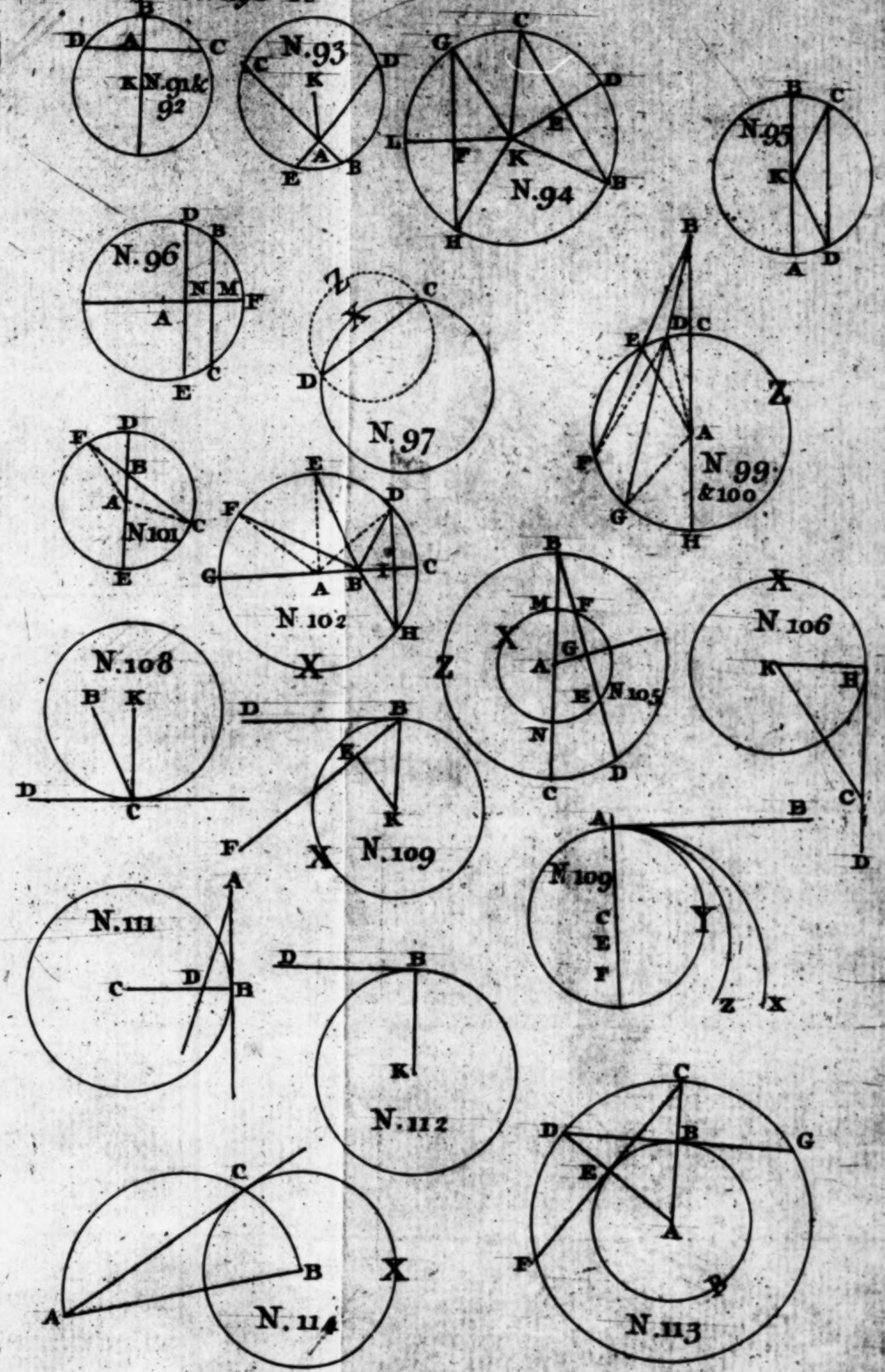
The Circle is BEB, the given Point is C, from which to the Center A draw a Line, and at the Point B where that Line interſects the Circle, by the preceedent Problem, draw the Tangent GD, thro' C deſcribe a concentric Circle, and from D where that Circle is cut by the Tangent GD, take DF equal to DC, and join C and F by a Line, which will be the Tangent required. By Conſtruction the Chord GD=CF, for the Arc GC is equal to the Arc CD,* and the Arc CD to the Arc DF, therefore the Arcs GD and CF being equal, their Chords are equal.† From D to the Center A draw the Line AD, which will be Perpendicular upon CF, ſeeing two of its Points, (*viz.*) A and D, are equally diſtant from its Ends, now ſince the Chords DG and CF are equal, the Lines AB and AE are equal,‡ then the Point E as well as B, is in the Circumference of the Circle BEB, therefore the Line CF being Perpendicular upon E, the End of the Radius AE, it touches the Circle.§

Or *more eaſily thus, let A be the given Point, from which a Tangent muſt be drawn to the Circle X, after having drawn the Line AB, from A to B, the Center of the Circle X, the Semi-circle ABC muſt be deſcribed upon that Line, and at C the Point of Section, draw AC which will be the Tangent required, the Demonſtration of which is not proper here.*

† *Sup. n.* 47.
‡ *Sup. n.* 106.
* *Sup. n.* 92.
† *Sup. n.* 31.
‡ *Sup. n.* 94.
§ *Sup. n.* 106.

⁂

Book 1. Plate 2. Page 26



BOOK. II.

Of Breadth, which is the ſecond Species of Extension.

Of Plane Surfaces.

SECT. I.

Of *Angles, or Surfaces which are between two Lines, that meet indirectly.*

ADVERTISEMENT.

In treating of Surfaces, we ſhall here conſider only Planes; that is to ſay, ſuch Surfaces as are the ſhorteſt between two right Lines; and ſhall begin firſt with thoſe which are comprehended between two Lines, that meet or interſect in a Point.

Definition 1. An *Angle is the aperture or opening of two Lines, which meet indirectly.* 1

Two Lines which meet directly, degenerate into only one Line.

Def. 2. The *Point where two Lines meet which form an Angle, is called the Vertex thereof, as is the Point B, and the two Lines are called the Sides, or Legs of the Angle.* 2

Thus, AB and BC are the Sides or Legs of the Angle ABC, when an Angle is marked with three Letters as ABC, the middlemoſt B denotes the Vertex, and the other two A and C, the Sides.

Def. 3. An *Angle which is upon a Plane, is called a plane Angle.* 3

Def. 4. There *are three Sorts of Angles, ſo eſteemed by reaſon of their Sides, the Rectilinear, Curvilinear, and Mixtilinear, the Rectilinear, is formed by the meeting of two right Lines, as C.* 4

The Curvilinear, is formed by the meeting of two Curve Lines, as A. The Mixtilinear, is formed by the meeting of a right Line and a Curve, as B.

Evident Propoſitions concerning Angles.

Proposition 1. From *the Definition of an Angle, it is evident, that its Bigneſs does not depend upon the Length of the Lines which form it, but upon their opening, or diſtance from each other.* 5

For whether the Lines AB and AC are long or ſhort, the opening is always the ſame, and the Space, or Surface which is between them, that is to ſay, which is neareſt the Vertex A, will be neither augmented or diminiſhed.

6 Prop. 2. An *Angle cannot be either augmented or diminiſhed, except one of its Sides in turning about the Vertex as upon a Center, either declines from, or inclines to, the other Side.*

7 Prop. 3. One *of the Sides of an Angle in turning round, and declining from the other Side, always makes the Angle greater, untill it becomes a right Line with the other Side, and then the Angle vaniſhes.*

8 Prop. 4. One *of the Sides of an Angle in turning about, can deſcribe only one whole Round or Circle, after which, it joins to the other Side, and makes therewith only one Line.*

9 Prop. 5. The *Arcs or Portions of different Circles, deſcribed through different Points of the Sides of an Angle, contain an equal Number of Degrees.*

The Point X of the Side AB or AC, deſcribes the Circle XX, and the Point Z the Circle ZZ, I ſay, that the Portions of the two Circles contained between the Radii AB and AC, are of an equal Number of Degrees. For ſince that they are deſcribed in the ſame Time, if we divide the Time into 360 Moments, the Circle having ſo many Degrees, in the firſt Moment, whilſt Z deſcribes the 360th Part of its Whole, it is evident, that X will alſo paſs through the ſame Part of the Circle which it deſcribes, for as the two whole Circles are compleated in the ſame Time, each Part is likewiſe made proportionably.

10 Prop. 6. *A Circle being drawn from the Vertex of an Angle as a Center, the Portion of the Circle, contained between the Sides of the Angle, is the meaſure or content thereof.*

11 Prop. 7. The *greateſt Angle that is, cannot have all the Semicircumference for its meaſure.*

For when the Leg AC of the Angle BAC, hath been turned about ſo as to make the Semi-circumference BCD, and that C is come to the Point D, it then makes only a right Line with AB: For ſince it is ſuppoſed, that BCD is the Semi-circumference, it muſt be, that AD+AB is the Diameter of the Circle, and that therefore, AD and AB are only one right Line, which divides the Circle into two equal Parts, the whole Circumference of the Circle is 360 Degrees, and conſequently, the Semi-circumference is 180 Degrees; therefore an Angle cannot contain 180 Degrees, for when the Side AB is come to D, AC and AB make only one right Line.

ADVERTISEMENT.

If AC paſſeth beyond DA and goes to E, it will by this Means, form the Angle BAE greater than 180 Degrees; which ſome Geometers call a reverſe Angle, and which is meaſured by the Arc BCDE, this Sort of Angle is not here conſidered, but only that of EAB, meaſured by the Arc EB, the Complement of the Arc BCDE to the whole Circle, to the Explanation of which we proceed.

OF the different Species of Angles with relation to their opening, or proportion to a Circle.

ADVERTISEMENT.

THERE are three Sorts of Angles, with reſpect to their opening or proportion to a Circle, which are, the right Angle, the acute Angle, and the obtuſe Angle.

DEF. 1. AN *Angle that is meaſured by half the Semi-circumference, or fourth Part of the whole Circumference of the Circle, that is, an Arc of ninety Degrees, is called a right Angle.* 13

THUS, ſuppoſing that the Arc AC is the fourth Part of the Circumference ACDE, and conſequently ninety Degrees, which is the fourth Part of 360; which make the whole Circle,* the Angle ABC, which is meaſured by the Arc AC, is a *right Angle.*

DEF. 2. AN *Angle that is meaſured by an Arc of more than ninety Degrees, is ſaid to be obtuſe.* 13

THE Angle FBC is obtuſe, the Arc FC which meaſures it, being more than ninety Degrees; becauſe it is greater than AC, the fourth Part of the Circle AFEDC.

DEF. 3. AN *Angle which is meaſured by an Arc leſs than ninety Degrees, is called Acute.* 14

THE Angle FBE being meaſured by the Arc FE, which is leſs than the Arc AE of ninety Degrees, is Acute. The acute Angle may be diminiſhed, *ad inf.* But the obtuſe Angle can be augmented only 'till it approaches infinitely near to two right Angles.

DEF. 4. THE *acute Angle, which together with the Obtuſe is equal to two right ones, is called the Complement of the obtuſe Angle to a Semi-circle.* 14

THUS, the Angle FBE, is the Complement of the obtuſe Angle CBF.

THEOREM 1. *A Line which is Perpendicular upon another Line, makes therewith two right Angles; and if it makes therewith two right Angles, it is Perpendicular.* 16

1. BA is Perpendicular upon A the Middle of CD, from which as a Center, having deſcribed the Semi-circle CBD, according to the Definition of a Perpendicular, the Lines or Chords BC and BD, are equal,* therefore the Arcs which they ſubtend, are equal,† and therefore ſince CBD is half of the Circumference, the Diameter of the Circle being CD, the Arcs BC and BD are Quadrants, then the Angles BAC and BAD are right ones, each being meaſured by the fourth Part of the Circle.¶ 2. It is eaſy to demonſtrate the ſecond Part hereof; for if the two Angles CAB and DAB are right ones, the Arcs BC and BD are equal, each half the Semi-circumference; ‖ but A and B, being therefore equally diſtant from C and D, the Line AB is Perpendicular.‡

THEOREM 2. EVERY *Line let fall upon another Line, makes therewith two Angles, equal to two right Angles.* Eucl. 1. Prop. 13. 17

---

* *B.* 1. *n.* 27.

* *B.* 1. *n.* 50. † *B.* 1. *n.* 31. ¶ *Sup. n.* 12. ‖ *Sup. n.* 12.

‡ *B.* 1, *n.* 43.

LET the Line be EA, which falls upon the Line DC, I ſay, that the Angle DAE together with the Angle EAC, is equal to two right Angles: For having from the Point A as a Center, deſcribed the Semi-circle DEC, and raiſed the Perpendicular AB, whether the Angle DAE be right, obtuſe, or acute, it is (*according to the Definitions* ¶) meaſured by the Arc DE, and the Angle EAC by the Arc EC,† theſe two Arcs together make the Semi-circumference of the Circle, which is equal to 180 Degrees, or two right Angles.§

18 CORALLARY. 1. IT *is evident, that how many Lines ſoever there be, which fall upon another Line at the ſame Point, they will form Angles, which altogether are equal to only two right ones.*

19 CORALLARY 2. THEREFORE, *two or more Lines, which interſect at a Point, being prolonged will form Angles, which altogether are equal to only four right Angles.*

LET as many Lines you pleaſe, be conceived to fall upon CD at the Point A, having from the Point A as a Center deſcribed a Circle, the Angles will be all meaſured by the Semi-circumference CGBD, which is the meaſure of two right Angles, having prolonged the Sides AB and AG, the Angles which they make are alſo equal to two right Angles, therefore all the Angles which can be made about the Point A, are equal to four right Angles.

20 THEOREM. 3. IF *two right Lines meet at a Point of any right Line, making on both ſides thereof, two Angles, equal to two right Angles, the two Lines meet directly.* Eucl. 1. Prop 14.

THE two Lines AB and AC, meet at the Point A of the right Line AE, and make on both ſides that Line, the two Angles EAB and EAC, equal to two right ones. It muſt be proved, that they meet directly, that is, being taken together they make only one right Line. $BAE+CAE$ are equal to two right Angles. If it is ſaid that BA and AC make two Lines, and that BA being prolonged goes to D, then by the preceeding Theorem, the Angles BAE and EAD, are equal to two right Angles; then $EAD=EAC$, *which is abſurd.*

21 THEOREM 4. THE *Vertical Angles of two Lines which interſect each other, are equal.* Eucl. 1. Prop. 15.

THE two Lines BC and ED interſect at the Point A. I ſay, that the Angles DAC and BAE are equal, as alſo DAB and CAE, BAD and DAC are equal to two right Angles; DAB and EAB make alſo two right Angles:¶ Therefore $BAD+DAC=BAD+EAB$, taking away from the two equal Quantities, the common Angle BAD, the remainders EAD and DAC are equal, the ſame reaſoning will prove that $BAD=EAC$.

22 DEF. 4. THE *Sine of an Arc is equal to half the Chord of twice that Arc, or to the Perpendicular let fall from the End of that Arc upon Radius.*

¶ *Sup. n.* 12, 13, *and* 14.

† *Sup. n.* 10, § *Sup. n.* 12.

¶ *Sup. n.* 17.

THE Arc EB is double the Arc BD, the Line BC, the half of BE, the Chord of BDE, is the Sine of the Arc BD, and alſo of the Arc BF, its complement to a Semi-circle, therefore the Arcs DE and BE, which together are equal to a Semi-circle, have the ſame Sine, and are complements to a Semi-circle, reciprocally each to each.

DEF. 5. THE *Sine of an Angle, is the Sine of the Arc that meaſures it.* 23

THUS BC which is the Sine of the Arc BD, the Meaſure of the Angle BAD, is the Sine of that Angle. When an Angle is obtuſe, its Sine is alſo the Sine of the Acute Angle, which is its complement to a Semi-circle, thus BC is both the Sine of the obtuſe Angle BAF, and of the acute Angle BAD, but 'tis only the Sines of the acute Angles, that are conſidered in the following Work, except the other are expreſsly mentioned.

BC being the Sine of the Angle BAD, the Line CD, contained between BC and the Arc BD, is called the verſed Sine of that Arc BD, therefore BC is called the right Sine, or ſimply the Sine, CD being always diſtinguiſhed by the Name verſed Sine, the Line DE which touches the Arc ED, and which is bounded by AE and AD, which comprehend the ſame Arc; it called the Tangent of both that Arc, and alſo of the Angle BAD, the Line AE is called Secant, both of that Angle, and alſo of that Arc BD.

THEOR. 5. EQUAL *Angles have equal Sines, and if the Sines are equal, the Angles are equal.* 24

1. THE Angles GEF and CAB are equal, therefore having from A and E their Vertex, with an equal Extent deſcribed the Arcs CB and GF, which meaſure theſe Angles, theſe Arcs are equal,* continue them ſo that GF=FN, and CB=BM, then ſince that equal Arcs have equal Chords,† CM=GN, and conſequently CO and GP, the Halves of theſe Chords, are equal, now theſe Halves are the Sines of the Angles CAB and GEF, according to the Definition, ‡ therefore the Sines of thoſe Angles are equal. 2. If the Sines CO and GP are equal, CM=GN,§ and therefore CBM=GFN, then the Angles CAB and GBF, being meaſured by the Halves of thoſe equal Arcs, they are equal.

THEOR. 6. IF *a Line cuts two parallel Lines obliquely, it forms eight Angles, four of which, are alternate interior ones, and four Exterior. I ſay, the Alternates, either Interior, or Exterior, are equal.* Ecul. 1. Prop. 29. 25

IT muſt be proved, that AFE=HGD, and AFG=FGD, the two firſt, are alternate Exterior, and the two others, alternate Interior. Between the two Parallels, draw the Perpendiculars FK and GI, which are equal, ‖ from F and G as Centers, with the diſtance GF, draw the Arcs AG and DF, which meaſure the Angles AFG and FGD, the Sines of theſe Arcs, or Angles, are the equal Perpendiculars GI and FK; * theſe Angles then are equal, † and AFE and AFG make two right Angles, ‡ as do FGD and DGH, therefore AFE+AFG=FGD+DGH, taking from each part, the equal Angles AFG and FGD, there will remain AFE=DGH.

---

* *Sup. n.* 10. † *b.* 1. *n.* 31. ‡ *Sup. n.* 22.
§ *Sup. n.* 22. ‖ *b.* 1. *n.* 65.

26 THEOREM. 7. IF *a Line which joins two other Lines, makes the Alternate Angles equal, the two Lines are Parallel.* Eucl. 1. Prop. 27.

IF AFG and FGD are equal; their Sines GI and FK are equal, § GI is Perpendicular upon AB, and FK upon CD, therefore by the Definition of Sines, || the two Lines AB and CD, * are Parallel.

27 THEOREM. 8. IF *a Line cuts two or more Parallel Lines, all the Angles that it makes with them taken on the ſame Side, are equal.*

IT is to be proved, that GAY=ABX=BCZ, that YAB=XBC =ZCH, and that on the other Side it is the ſame; that GAF= ABE=BCI, and FAB=EBC=ICH. 1. ZCB=CBE, * and CBE=ABX, † then ZCB=CBE=XBA; therefore according to this Axiom, *that two Quantities equal to a Third are equal.* ZCB =XBA; In the ſame Manner it is demonſtrated, that GAY= ABX &c.

28 THEOREM. 9. IF *a Line that falls upon ſeveral other Lines, makes therewith, the Angles taken on the ſame Side, equal, the Lines are Parallel.* Eucl. 1. Prop. 28.

GAY+YAB is equal to two right Angles, as is alſo GBX+ GBE, ‡ then taking away from the two equal wholes, the Angles GAY and GBX, which are ſuppoſed equal, the Alternate Angles YAB and GBE, will remain equal. Therefore the Lines Y and Z are Parallel, § It is demonſtrated in the ſame Manner, that Y and Z, or X and Z are Parallel.

29 PROBLEM. 1. FROM *a Point given in a right Line, to deſcribe a Rectilinear Angle, equal to an Angle given.* Eucl. 1. Prop. 2.

FROM the given Point A of the right Line Z, it is required to deſcribe an Angle CAB, equal to the Angle EDF, from the Point D as a Center, draw the Arc EF, then from the given Point A as a Center, with the extent of DE, draw the Arc X, from which take the Arc BC, equal to the Arc EF, by means of equal Chords EF, BC, || then drawing a right Line from C to the Point A, the Angle CAB, will the Angle required, equal to EDF, for they are meaſured by equal Arcs, therefore they are equal *.

30 PROBLEM. 2. FROM *a Point given out of a Line, to draw a right Line upon another Line, ſo as to make therewith an Angle, equal to an Angle given.*

LET

* *Sup. n.* 23. † *Sup. n.* 24. ‡ *Sup. n.* 17.
§. *Sup. n.* 24. || *Sup. n.* 22.

* *b.* 1. *n.* 71. † *Sup. n.* 25. ‡ *Sup. n.* 21.
§ *Sup. n.* 17, || *Sup. n.* 26.

* *b.* 1. *n* 3. * *Sup. n.* 10.

Let the given Point be D, from which muſt be drawn upon Z, a right Line making therewith an Angle, equal to X. From any Point of Z taken at Pleaſure, draw a Line by the preceeding Problem, making therewith the Angle CAB, equal to the given Angle X, ſuch is AC, if this Line paſs thro' the Point D, the Problem is done. If it doth not paſs thro' D, draw a Line thro' D, parallel to CA, † then DEB=CAB=X, ‡ therefore DEB =X.

Problem. 3. To *biſect, or divide a given Angle, into two equal Parts.* Eucl. 1. Prop. 9. 31

The given Angle is BAC; having made the two Sides AB and AC, equal, join them by the Line BC, upon which having drawn a Perpendicular AG, § from the Point A. BD=DC, ‖ therefore the Arc BGC, being conceived as part of a Circle whoſe Center is A. AB and AC the Radii thereof, and BC the Chord which is cut equally in two by the ſaid Perpendicular AG, as is alſo the Arc BC at the Point G. * therefore the Arc BG is equal to the Arc GC, and therefore the Angles BAG, GAC which they meaſure are equal, † the Angle BAC is then cut equally in two.

Theorem. 10. The *Angle comprehended between a Circle and its Tangent, is leſſer than any rectilinear Angle whatſoever,* Eucl. 3. Prop. 16. 32

Between the Circle and its Tangent cannot be drawn a right Line, ‡ therefore the mixt Angle made by the Circle and its Tangent, cannot be divided; this Angle is therefore leſſer than any rectilinear Angle whatſoever.

SECTION. II.

Of the Compariſon of Angles, and of their different Poſition with reſpect to a Circle.

ADVERTISEMENT.

An Angle, as has been aforeſaid, is meaſured by the Arc of a Circle whoſe Center is the Vertex of the ſaid Angle, and which is bounded by the Sides which form it. Now the Vertex of an Angle may be either at the Circle's Center, or out of the Center, in the Circumference, or out of the Circumference; and altho' the Angle may in all theſe caſes be the ſame, yet it affords different conſiderations.

Definition. 1. The *Angle whoſe vertex is in the Circumference of the Circle, and the Sides in the Circle, is called an Angle at the Circumference, as* BAC. 33

† *b.* 1. *n.* 72. ‡ *Sup. n.* 27. § *b.* 1. *n.* 46.
‖ *b.* 1. *n,* 42.
* *b.* 1. *n.* 84. † *Sup. n.* 10. ‡ *b.* 1. *n.* 109.

Euclid calls thoſe inſcribed Angles, which are in the Circumference; and thoſe Circumſcribing ones which are out of the Circle, but which touch it with their Sides; BAC is an inſcribed Angle, and EDF a Circumſcribing one.

34 Def. 2. An *Angle at the Center, is an Angle whoſe Vertex is at the Center of the Circle, and whoſe ſides are Radius's, as O M N.*

35 Def. 3. A *Segment of a Circle, is that part of the Circle which is included between the Circumference, and a Chord Line, thus PR divides the Circle into a greater, and a leſſer Segment.*

36 Def. 4. The *Angle made by a Tangent and a Chord Line, or by a Secant, drawn from the Point of contact, is called an Angle of the Segment, as QPR or RPT.*

37 Theorem. 1. The *Angle of the Segment is meaſured by half the Arc that is bounded by its Chord.* Eucl. 3. Prop. 32.

Let the Angle be CBE, made by the Tangent CB, and the Chord BE, it muſt be proved, that its meaſure is half the Arc BE. Let there be drawn the Diameter GH cutting the Arc and the Chord EB into two equal Parts; and draw alſo the Diameter DF Parallel to the Chord EB, and the Radius AB paſſing thro' the Point of contact, will form an Angle at the Center GAB, which is meaſured by the Arc GB. It remains then only to prove that the Angle CBE is equal thereto, which is evident; for the Angle CBA is a right one as well as the Angle GAF; * but the Angle EBA is equal to the Angle BAF, becauſe they are Alternate ones, † then taking away the two equal Angles, *viz.* FAB from GAE which is a right Angle, and ABE from the right Angle ABC, the Remainders BAG and CBE are equal. Now BAG is meaſured by the Arc BG; then CBE which is equal thereto, is meaſured by an Arc equal to BG, the half of BGE; *which was to be proved,* The ſame Way of reaſoning will prove, that the other Angle DBE is meaſured by the Arc BH, the half of the Arc BHE, comprehended in the ſaid Angle between the Tangent BD and the Chord BE, by adding to the right Angles DBA, FAH, the equal alternate Angles EBA, BAF, inſtead of Subtracting as in the preceeding Demonſtration.

38 Corallary. It *is evident, that if an infinite Number of Arcs be drawn between the Tangent and the Chord, and which are croſſed by producing the Chord, they are all of an equal Number of Degrees, ſeeing they are the Meaſure of one and the ſame Angle.*

For the Angle BAC being meaſured by the half of AD, of AE, of AF, &c. If this Angle for Example contain'd ten Degrees, all theſe Arcs will be each twenty Degrees.

39 Theorem. 2. The *Angle at the Circumference, is meaſured by half the Arc which it ſtands upon.*

Let the Angle be BAC. It muſt be proved that it is meaſured by half the Arc BC. Thro' the Point A let there be drawn the Tangent ED; which will Form two new Angles EAB and DAC, which by the preceeding Theorem are meaſured by half the Arcs AB, AC, but the three Angles formed at the Point A are equal to two right Ones: * therefore they are meaſured by half the Circumference of the Circle, † then the halves of the Arcs AB and AC being theMeaſures of the Angles BAE and CAD, half the remainder of the Circle, that is, the half of BC, is the meaſure of BAC.

---

* b. 1, n. 91 & 103. † *Sup. n.* 25.

40 CORALLARY. 1. IT *is evident, that all the Angles at the Circumference, and which ſtand upon the ſame Arc, are equal.* Eucl. 3. Prop. 21.

THUS the Angles ABC and ADC, in any part of the Circumference ABDC whatſoever, are both equal; for they are both meaſured by half the Arc AC, upon which they ſtand: therefore having one and the ſame meaſure, they are equal.

41 CORALLARY. 2. AN *Angle at the Center, is double the Angle at the Circumference which ſtands upon the ſame Arc.* Eucl. 3. Prop. 20.

LET the Angle at the Center be CAD, and that at the Circumference CBD; it is plain, that the firſt whoſe meaſure is the whole Arc CD is double CBD, whoſe meaſure is only the half thereof.

42 IN *equal Circles, equal Angles ſtand upon equal Arcs, whether they be at the Center or at the Circumference,* Eucl. 3. Prop. 26.

IF it were otherwiſe, the Angles having unequal meaſures, they would be unequal; whereas they are ſuppoſed to be equal.

43 CORALLARY. 4. IN *equal Circles, the Angles (whether at the Center or at the Circumference) ſtanding upon equal Arcs, are Equal.* Eucl. 3. Prop. 27.

THEY have equal meaſures; therefore they are equal.

44 CORALLARY. 5. AN *Angle at the Circumference of a Semi-Circle, or whoſe Baſe is the Diameter of the Circle, is a right Angle.* Eucl. 3. Prop. 31.

FOR it ſtands upon the Semi-circumference, whoſe half, (which is Ninety degrees) is the Meaſure of a right Angle. ¶

45 CORALLARY. 6. AN *Angle in a Segment greater than a Semicircle, is Acute.* Eucl. 3. Prop. 31.

FOR it ſtands upon an Arc leſs than the Semi-circumference, therefore the half of that Arc, (which is its meaſure,) is leſs than ninety Degrees; therefore the Angle is Acute. §

46 CORALLARY. 7. AN *Angle in a leſſer Segment is obtuſe,* Eucl. 3. Prop. 31.

FOR it ſtands upon an Arc greater than the Semi-circumference, whoſe half (which is its meaſure,) is more than ninety Degrees; the Angle is therefore obtuſe. ‖

Corallary

---

* *Sup. n.* 18. † *Sup. n.* 12. ¶ *Sup. n.* 12.
§ *Sup. n.* 14. ‖ *Sup. n.* 13.

47 Corallary. 8. Angles *at the Circumference which are oppoſite, and placed upon the ſame Points, are equal to two right Angles.*

Let the Angles be ACD, and ABD, it is plain, that they are meaſured by half the Arcs ABD, and ACD; * and Conſequently by half the whole Circumference, which is equal to twice ninety Degrees; therefore the two Angles having together the Meaſure of two right Angles, are equal to two right Angles.

48 Problem. 1. To *cut off from a Circle, a Segment containing an Angle given.* Eucl. 3. Prop. 34.

Let the Circle be X from which muſt be taken a Segment ſufficient to contain an Angle equal to the given Angle A, or which is the ſame, that the Angle which is placed upon the other Segment may be equal to the given Angle A. Draw DF ſo as to touch the Circle X, † and upon DF draw another Line DE making with DF an Angle equal to the Angle A, ‡ every Angle inſcribed in the Circle X, and which ſtands upon DE, is meaſured by half the Arc ED. § Now half that Arc meaſures the Angle EDF, equal to A; ‖ therefore what was propoſed is done; *that is to ſay,* that every Angle inſcribed in the Circle X, whoſe Baſe is the Arc DE, (what part of the Circumference ſoever its Vertex is in,) will be equal to the Angle A.

49 Problem. 2. To *find the Circle whoſe Segment bounded by a given Line, will contain an Angle equal to an Angle given,* Eucl. 3. Prop. 33.

Upon BD make the Angle FBD equal to the Angle K, * from the Point B draw BC Perpendicular upon BF, † and upon the Middle of BD another Perpendicular EC, which will cut BC at the Point C; from whence having with the diſtance of BC deſcribed a Circle, ‡ it will be the Circle required, *which is to be proved.* BF is Perpendicular upon the Radius BC, and touches the Circle.§ The Angle FBD is meaſured by an Arc equal to half the Arc BD; ‖ all the Angles inſcribed in the Circle and which ſtand upon BD are equal, and are meaſured by half the Arc BD; * they are then equal to the Angle FBD; and therefore to the Angle K, to which FBD is equal by conſtruction.

ADVERTISEMENT.

From what has been proved, that all Angles ſtanding upon the ſame Arc are equal, † is deduced a very uſeful Method of deſcribing the part of a Circle containing as many Degrees you pleaſe, without having Compaſſes, or the Center of the Circle.

AB

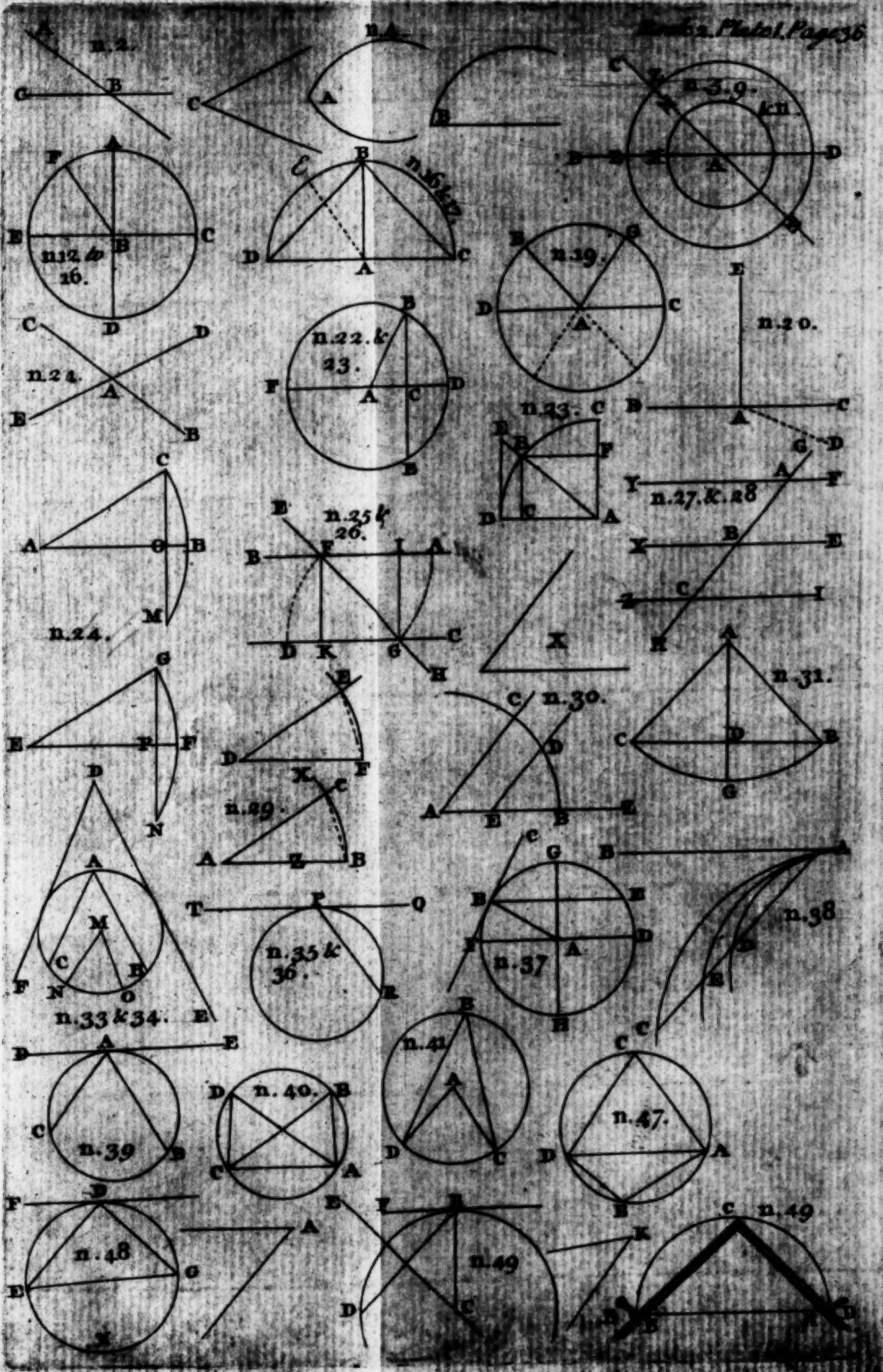
---

* *Sup. n.* 39. † *b.* 1. *n.* 112. ‡ *b. ſup. n.* 29.
§ *Sup. n.* 39. ‖ *Sup. n.* 37.

* *Sup. n.* 29. † *b.* 1. *n.* 47. ‡ *b.* 1. *n.* 30.
§ *b.* 1. *n.* 106. ‖ *Sup. n.* 37.

* *Sup. n.* 39. † *Sup. n.* 40.

Book 2. Plate 1. Page 36



AB is the Chord of an Arc propoſed, or portion of a Circle, required to be deſcribed. Admit the Arc be of ten Degrees, hence, the content of the Angle inſcribed in this Arc, is the half of 350 Degrees, or of three hundred and ſixty Degrees leſs ten, that is, this Angle is one hundred and ſeventy five Degrees; adjuſt the two ſtraight Rules CD and CE, ſo that the Angle DCE be of one hundred and ſeventy five Degrees, and join them together, fix two nails at A and B, the Ends of the given Chord Line, after which turning about the Point C, ſo that the two Rules, CD and CE may raze always upon the Nails A and B, they will thereby deſcribe the circular Line ACB, which is the Arc required. The part of a Circle of any bigneſs required, may be this way deſcribed.

THE *above is a mechanical Operation, but the following is Geometrical.*

50 PROBLEM. 3. THE *Chord of a Segment and the Angle in the Segment being given, to find the Points thro' which the Arc paſſeth whoſe Chord is given, without knowing or finding the Center of the Circle, of which that Arc is a part.*

THE given Chord of the Segment required to be deſcribed is AB, draw the Line BD, making any Angle with BA, then at a Point in that Line taken at Pleaſure, make the Angle ECF equal to the Angle given; then thro' A draw a Line parallel to EC; then the Angle AFB is equal to ECF * and alſo to the Angle given. Wherefore according to what has been demonſtrated, the propoſed Arc paſſes thro' F. In like manner, the other Points are found thro' which that Arc paſſeth, without finding the Center of the Circle of which the Arc is a part.

51 THEOREM 3. AN *Angle at the Circumference, made by a Chord and a Secant prolonged out of the Circle, is meaſured by half the Arc of the Chord, more half the Arc of the Secant.*

LET AB be a Chord, and CD a Secant prolonged out of the Circle, one part of which is the Chord of the Arc CA. It muſt be proved, that the Angle BAD formed by the Chord AB, and the Secant CD, is meaſured by half the Arc AB, more half the Arc AC. The two Angles CAB and BAD are together equal to two right Angles; † they are then meaſured by the Semicircumference of the Circle; ‡ but the Angle CAB is meaſured by half the Arc CB: § The remaining Angle BAD will then be meaſured by half the remaining Arcs CA, AB, which together with the Arc CB make the whole Circle.

52 THEOREM 4. THE *Angle formed by the meeting of two Chords which interſect within a Circle, is meaſured by the Sum of half the Arcs upon which it ſtands.*

LET B be the Vertex of the Angle within the Circle. It muſt be proved, that its meaſure is half the Arcs AE and DC. If B was the Center, this would be evident. As the Point G is the Center, thro' G draw Parallels to the Sides of the Angle CBD, the Angle CBD is equal to the Angle IGH, * whoſe meaſure is the Arc HI. It is then required only to prove, that the Arc HI is half the Arcs DC and AE. The Arcs HI and FL are equal.† $DI=AL$ and $CH=EF$. ‡ Now $HI=AE+AL+EF$. As alſo $HI=DC-CH$ (or EF) $-DI$ (or AL), *that is to ſay*, $HI=DC-EF-AL$. Conſequently $HI+HI=AE+AL+EF+CD-EF-AL$. Now $+AL+EF-AL-EF=$Cypher, then $HI+HI$ (or $2\,HI$) $=AE+DC$. Conſequently the Arc HI is alſo half the Arcs DC and AE; which are therefore the Meaſure of the Angle CBD equal to IGH, whoſe meaſure is the Arc HI.

---

* *Sup. n.* 27. † *Sup. n.* 17. ‡ *Sup. n.* 12.
§ *Sup. n.* 39.

53 THEOREM. 5. AN *Angle whoſe Vertex is without the Circle, but whoſe Sides croſs it, and ſtand upon the Circumference, is meaſured by half the concave Arc leſs half the convex Arc.*

LET the Angle be BAC. I ſay that it is meaſured by half the Arc BC leſs half the Arc DF. Thro' D draw a Line parallel to AC; § therefore the Angle BAC is equal to BDE, ‖ which is meaſured by half the Arc BE. * The Arcs CE and DF are equal. † Now $BE=BC-CE$; then $BE=BC-DF$. Then the half of BE is equal to half $BC-DF$, therefore ſince the Angle BAC is meaſured by half BE; its meaſure is half the concave Arc BC, upon which it ſtands, leſs half the convex Arc DF; *which was to be proved.*

54 THEOREM. 6. AN *Angle whoſe Sides touch the Circle, is meaſured by half the concave Arc leſs half the convex Arc.*

LET the Angle be BAC, whoſe ſides AB, AC, touch the Circle. It is to be proved, that its meaſure is half BEC leſs half BFC. Thro' C one of the Points of contact, draw CE parallel to AB, ‡ the Angle BAC is equal to the Angle DCE. § Now DCE is meaſured by half the Arc CE, ‖ which alſo meaſures CAB: It remains therefore to prove, that the Arc $BEC-BFC$ is equal to the Arc CE. 1. The Arc $BEC-BE$ is equal to the Arc EC. 2. Since the Arc BC * is equal to the Arc BE; then $BEC-BFC=EC$: *which was to be proved.*

---

* *Sup. n.* 27. † *Sup. n.* 21. *and n.* 42. ‡ *b.* 1. *n.* 86.
§ *b.* 1. *n.* 72. ‖ *Sup. n.* 27.

* *Sup. n.* 39. † *b.* 1. *n.* 80. ‡ *b.* 1. *n.* 72.
§ *Sup. n.* 27. ‖ *Sup. n.* 37.

* *b.* 1. *n.* 36.

SECT. III.

Of TRIANGLES

DEFINITION. I.

A Triangle is a Surface bounded by three Lines. There are ſix 55
*Sorts thereof: Three in reſpect of their Sides, and three with
regard to their Angles.*

DEF. 2. *A Triangle whoſe three Sides are equal is called Equi-* 56
lateral, as ABC.

DEF. 3. *A Triangle that has only two Sides equal is called* 57
Iſoceles, as HGI.

DEF. 4. *A Triangle whoſe Sides are all unequal is called Sca-* 58
lene, as DEF.

DEF. 5. *A Triangle which hath one right Angle, is called a* 59
Rectangled Triangle, as LKM.

DEF. 6. *A Triangle that hath one obtuſe Angle, is called Ob-* 60
tuſe-angled, or Ambligonium, as NPO.

DEF. 7. *A Triangle whoſe Angles are all Acute, is called* 61
Acute angled, or Oxygonium, as ABC.

DEF. 8. *A Triangle is ſaid to be inſcribed in a Circle, when* 62
*its three Angular Points are in the Circumference thereof, and the
Circle is then ſaid to circumſcribe the Triangle.*

DEF. 9. *A Triangle is ſaid to circumſcribe a Circle, when all* 63
*its Angular Points are without the Circle, and whoſe Sides touch the
Circle; and then the Circle is ſaid to be inſcribed in the Triangle.*

DEF. 10. Two *Triangles are Equiangular, when the Angles of* 64
one are equal to the Angles of the other, each to each.

DEF. 11. Two *Triangles are ſaid to be entirely equal, which* 65
*being Equiangular, and the Sides containing the Angles of one equal
to the Sides containing the Angles of the other, each to each.*

THEOREM 1. ANY *two Sides of a Triangle, are together longer* 66
than the third. Eucl. 1. Prop. 20.

$AB+BC$ is longer than AC; It has been already aſſerted, that between the two Points A and C, there cannot be any Line conceived ſhorter than the right Line AC. *

CORALLARY. THEREFORE *a Triangle cannot be made of three* 67
given Lines, unleſs two of them are together longer than the third,
Eucl. 1. Prop. 22.

PROBLEM. 1. THREE *Lines being given, to make thereof a* 68
Triangle, Eucl. 1. Prop. 1. and 22.

LET the three given Lines be A, B, C. from the Ends of one of the three Lines, as C for Example, with the Extent of the Line

* *b.* 1. *n.* 12.

Line A deſcribe an Arc, and from the other End with the Extent of B draw another Arc; then having drawn from the Point where theſe two Arcs interſect, the two Lines A and B to the Ends of C, they will make the Triangle required, whoſe Sides are by conſtruction equal to the given Lines.

69 **Problem. 2.** To *deſcribe an equilateral Triangle upon a Line given.*

Let the given Line be AB, with the extent of this Line, and from the Ends A and B as Centers, deſcribe two Arcs, from the Point where they interſect, draw Lines thro' A and B, and they will make an Equilateral Triangle.

70 **Problem. 3.** To *circumſcribe a Circle about a given Triangle.* Eucl. 4. Prop. 5.

Is the ſame thing as to make a Circle paſs thro' three given Points.*

71 **Corallary.** *It is evident that to find a Point equally diſtant from three given Points, which are not upon the ſame Line, a Circle muſt be deſcribed thro' thoſe three Points; the Center of that Circle is the Point required.*

72 **Def. 12.** In *every Triangle, the Angle formed by producing one of its Sides, is called an exterior Angle.*

In the Triangle BAC, having produced BC to D, the Angle ACD is called an exterior Angle: being conſidered as oppoſite to the two Interior ones CAB and ABC.

73 **Theorem. 2.** The *exterior Angle of a Triangle is equal to the two oppoſite Interior ones.* Eucl. 1. Prop. 32.

Draw CE parallel to AB, and the exterior Angle ACD is thereby divided into two Angles ACE and ECD, which muſt be proved equal to the two oppoſite Interior ones, which is evident; for ABC=ECD, † and BAC=ACE. ‡

74 **Corallary.** The *exterior Angle is therefore greater than either of the oppoſite Interior ones.* Eucl. 1. Prop. 16.

This muſt certainly be ſo, by reaſon the exterior Angle is equal to both the Interior oppoſite ones, as has been demonſtrated.

75 **Theorem. 3.** The *three Angles of a Triangle are together equal to two right Angles.* Eucl. 1. Prop. 32.

To demonſtrate which, circumſcribe a Circle about the given Triangle. § The three Angles of the Triangle are meaſured by half the Arcs which they ſubtend, ‖ therefore they are meaſured by half the Circumference of the Circle, *that is* 180 Degrees, the Sum of two right Angles. *

Or thus. The two Angles ACB and ACD are equal to two right Angles; † but ACD the exterior Angle, is equal to the two oppoſite Interior ones ABC and CAB, ‡ therefore theſe two Angles together with ABC are equal to two right Angles.

---

* *b.* 1. *n.* 87. † *Sup. n.* 27. ‡ *Sup. n.* 25.
§ *Sup. n.* 70. ‖ *Sup. n.* 39.
* *Sup. n.* 12. † *Sup. n.* 17. ‡ *Sup. n.* 73.

Corallary. 1. Therefore *the two Angles of a Triangle being known, the third is alſo known.* 76

For the three together are equal to 180 Degrees, if two of them make 160, the third ought to make 20.

Corallary. 2. The *three Angles of a Triangle may be all Acute.* 77

For one hundred and eighty Degrees, which is the Sum of the three Angles of a Triangle, may be divided into three parts, and each of them will be leſs than the Sum of either a right, or an obtuſe Angle, and which all together make only one hundred and eighty Degrees, the Sum of two right Angles.

Corallary. 3. In *a Triangle there can be but one right Angle, nor but one obtuſe Angle.* 78

If there were two right Angles, the three would together make more than two right Angles. If two were obtuſe, the obtuſe Angle being greater than the right Angle, the three together would alſo be more than two right ones; which is contrary to the preceeding Theorem.

Corallary. 4. Two *of the Angles of a Triangle muſt therefore be Acute, or leſs than two right Angles.* Eucl. 1. Prop. 17. 79

This is evident, ſince that a Triangle cannot have two of its Angles, either Right, or Obtuſe.

Corallary. 5. If *in two Triangles, two Angles of one are equal to two Angles of the other, they are equiangular, that is, the third Angle of the one is equal to the third Angle of the other.* 80

Corallary. 6. If *a Triangle hath one of its Angles greater than that of another Triangle, the other two are together leſs than the other two of the ſaid Triangle.* 81

For a greater part being ſubtracted from the Sum of two right Angles, the remainder ought to be leſſer, than it would if a leſſer part had been taken away.

Theorem. 4. The *three Angles of a Scalene Triangle are unequal.* 82

Let ABC be a Scalene Triangle, circumſcribe it with the Circle X. * Since that the three Sides AB, AC, BC, are unequal, the three Arcs of which they are the Chords are unequal, † Conſequently the three Angles of the Triangle ABC, which are meaſured by half theſe Arcs, ‡ are unequal.

In *an Iſoceles Triangle, the Angles at the Baſe are equal; and if the Angles at the Baſe are equal; the Triangle is Iſoceles.* Eucl. 1. Prop. 5. 83

Let ABC be an Iſoceles Triangle, circumſcribe it with the Circle X, ſince that AB=AC, the Arcs ſubtended by thoſe Sides are equal. Now the Halves of theſe Arcs are the Meaſure of the Angles ABC and ACB; § theſe Angles are therefore equal. The other part of this Propoſition is eaſy. For if the two Angles

---

* *Sup. n.* 70. † *b.* 1. *n.* 32. ‡ *Sup. n.* 39.

§ *Sup. n.* 39.

ABC and ACB are equal, the Arcs AB and AC, or their Chords are equal; therefore the Triangle ABC is an Iſoceles.

84 CORALLARY. 1. THE *Angles at the Baſe of an Iſoceles Triangle, can neither of them be either Right, or Obtuſe.*

FOR if one be a Right one, the other would be the ſame alſo: thus the Sum of the three Angles would be more than two right Angles; which cannot be. * and if one was obtuſe, the other would be ſo likewiſe; ſo that only two Angles of the Triangle would make more than two right ones: *which is alſo impoſſible.*

85 CORALLARY 2. IF *either of the Vertical Angles, or one of the two Angles at the Baſe of an Iſoceles are Equal, their Angles are all equal.*

FOR if the equal Angle be at the Baſe, the other Angle at the Baſe will be alſo equal, and ſo likewiſe the third, † 2. If the equal Angle be at the Vertex, the Sum of the two Angles at the Baſe of each Triangle is the ſame; now each is half that ſame Sum, therefore they are equal.

86 THEOREM. 6. THE *three Angles of an Equilateral Triangle are equal.*

HAVING deſcrib'd a Circle about the Equilateral Triangle, the Sides of the Triangle being the Chords thereof, are conſequently equal, ‡ and their halves equal. § Now theſe halves meaſure the Angles of the Triangle. Therefore the Angles are all equal.

87 CORALLARY. HENCE *each Angle of an Equilateral Triangle is Acute, and always of ſixty Degrees.*

ADVERTISEMENT.

88 FROM this Corallary is deduced a different Method, for raiſing a Perpendicular from a given Point of a Line, for Example, from the Point A of the Line AB; make the Equilateral Triangle ABC, then produce BC to D, ſo that CD=CB; from D to A draw a right Line, which will be the Perpendicular, if BAD is a right Angle. Now I demonſtrate that it is; for the the Angle ACB is equal to the two oppoſite ones D and A which are equal, by reaſon CD is equal to CA. Hence ACB being 60 Degrees, DAC is 30, CAB is 60; therefore the Angle BAD is 60+30, or 90 Degrees; which is the content of a right Angle.

89 THEOREM. 7. IN *a Triangle, the longeſt Side ſubtends the greateſt Angle, and the greateſt Angle is ſubtended by the longeſt Side,* Eucl. 1. Prop. 18. and 19.

HAVING deſcribed a Circle about the Triangle, the longeſt Side of the Triangle will ſubtend the greateſt Arc. ‖ Now the half of that Arc is the Meaſure of the Angle oppoſite to the longeſt Side,* therefore that Angle is the greateſt, which is meaſured by half the greateſt Arc. If a Triangle be inſcribed in a Circle, the greateſt Angle is meaſured by half the greateſt Arc, now the greateſt Arc has the longeſt Chord ;† hence the Side oppoſite to that Angle is the longeſt.

---

* *Sup. n.* 75. † *Sup. n.* 80. ‡ *b.* 1. *n.* 31.
§ *Sup, n.* 39. ‖ *b.* 1. 32.

THEOREM 8. IF *two of the Angles of a Triangle are equal, the Sides oppoſite to thoſe Angles are equal.* Eucl. 1. Prop. 6. 90

HAVING inſcribed the Triangle in a Circle, the equal Angles are ſubtended by equal Arcs, having equal Chords, which are the Sides oppoſite to thoſe Angles.

THEOREM 9. IN *a Triangle, the half of each Side is the Sine of the oppoſite Angle,* 91

LET the Triangle ABC be inſcribed in the Circle X: It muſt be demonſtrated that AD, the half of AC, is the Sine of the Angle ABC. The Arc AE, which is the half of the Arc AC, meaſures the Angle ABC ;* hence the Arc AC is double the Arc which meaſures the Angle ABC; therefore AD, half the Chord of the Arc AC, is the Sine of the Arc AE,† and alſo that of the Angle ABC. The ſame way it is demonſtrated, that half BA is the Sine of the Angle ACB, and the half of BC that of BAC.

ADVERTISEMENT.

THEREFORE the Sine of one Angle, is to the Side oppoſite to that Angle, as the Sine of another Angle, is to its oppoſite Side: Or the Sines of Angles, are to each other as the oppoſite Sides, ſeeing that the Halves are as the Wholes.

THEOREM 10. TWO *Triangles whoſe Sides are equal, are Equi-angular, and entirely equal to each other.* 92

THE Sides of the Triangles ABC and DEF are equal, I ſay, that they are Equi-angular, and entirely equal, or (*which is the ſame thing*) that being placed one upon another, they will coincide. 1. BC being equal to DE, it is plain, that if the Line DE be placed upon BC, they will coincide together. If it is ſaid that DF will not coincide with AB, nor FE with AC, I prove the the contrary. From B as a Center, with the diſtance of the equal Lines AB or DF, deſcribe the Arc Z; and from C with the extent of AC or EF (which are equal,) draw the Arc X; theſe two Arcs muſt neceſſarily interſect as here at A. 2. It is evident that D being placed upon B, the Point F muſt neceſſarily be found in the Arc Z, and that E being placed upon C, the Point F, will be found in Z and X, and therefore in the Point A, where the two Circles interſect: Hence, the two Triangles being placed one upon the other, will every where agree; and are equal. Q. E. D.

IT may be ſaid that the two Arcs Z and X, will croſs in another Place beſide A, which is true, but by what has been demonſtrated, it can be only in two Points,‡ and the ſecond Point muſt

---

* *Sup. n.* 39. † *B.* 1. *n.* 32.

* *Sup. n.* 39. † *Sup.* 22 *and* 23. ‡ *B.* 1. *n.* 89.

be below BC (*viz.*) at the Point G, as is evident; hence, if they croſs each other above BC, in any other Point beſide A; they would interſect in three Points; *which cannot be.**

93 CORALLARY 1. IF *from the Ends of a right Line two other right Lines are drawn which meet at a Point, there cannot be drawn two other Lines from the ſame Ends, and on the ſame Side, equal to the two firſt, each to each, ſo as to meet in another Point.* Eucl. 1. Prop. 7.

FOR they make two Triangles having equal Sides, which conſequently being Equi-angular, ought to coincide together.

94 CORALLARY 2. TWO *Triangles having two Sides equal, each to each, and the Baſes equal, the Angles comprehended by the two equal Sides, are equal.* Eucl. 1. Prop. 8.

THE two Triangles having their Sides equal, are entirely Equi-angular.

95 THEOREM 11. TWO *Equi-angular Triangles, which have one Side equal, are entirely equal.*

ABC and DEF are Equi-angular, and AB=DE, I ſay, they are entirely equal, for if one is placed upon the other, DE will coincide with AB, ſince they are equal Lines. If DF does not coincide with AC, but with AG, as the Angles FDE and CAB are equal, it will follow that the Angle CAB=GAB, which is the ſame as FDE; which cannot be, it muſt therefore be that DF will coincide with AC, and by the ſame reaſon FE with BC, and the Point F with C, and conſequently the two Triangles are both every way equal.

96 CORALLARY. TWO *Triangles having two Angles and one Side equal, are entirely equal.* Eucl. 1. Prop. 26.

FOR two Triangles which have two Angles equal, are entirely Equi-angular,* conſequently if they have one Side equal, it muſt be, *according to the Theorem,* that they are entirely equal.

97 PROBLEM 4. TO *make a Triangle, having one Side, and the Angles at that Side given.*

THE given Side is AB, let the Angle at the Point A be called X, and that at B, be called Z. Upon A draw the Line AC, making the Angle CAB equal to X,† and upon B draw the Line BD, making the Angle ABD equal to Z; theſe two Lines will interſect, unleſs X and Z are either right Angles, or both together greater than two right Angles, for if they are, the Problem would be impoſſible,‡ and the two Lines would not interſect, but be parallel,§ or elſe decline from each other, and they muſt incline in order to make a Triangle ſuch as ABE, which is that which was propoſed to be made, for 1. Having two equal Angles, they are Equi-angular; ‖ and having one Side equal, they are entirely equal; by the preceeding Corallary.

98 THEOREM 12. TWO *Triangles that have one Angle equal, and the Sides containing that Angle equal, are wholly equal.* Eucl. 1. Prop. 4.

THE

* *B.* 1. *n.* 88. * *Sup. n.* 80. † *Sup. n.* 29.
‡ *Sup. n.* 79. § *B.* 1. *n.* 68. ‖ *Sup. n.* 80.

THE Angle BAC=EDF, and AB=DE, and AC=DF, I ſay, that the two Triangles being put one upon another will coincide; For DE will coincide with AB; if DF doth not agree with AC, but with AH, the Angle BAC will be equal to the Angle BAH, which cannot be: Therefore DF agrees with AC, and the Point F with C, conſequently BC=EF; therefore the two Triangles having their Sides equal, are Equi-angular,* *that is*, equal in every Part.

THEOREM 13. TWO *Triangles having one and the ſame Baſe, the Vertical Angle of the contained Triangle, is greater than the Vertical Angle of the containing Triangle.* Eucl. 1. Prop. 21. 99

THE two Triangles CAD and CBD, have CD for their Baſe. CAD is contained in CBD, it is to be proved, that $A > B$. Prolong DA to E, the exterior Angle CAD is equal to the two interior oppoſite ones AEC, ECA;† therefore it is greater than either of them, by the ſame reaſon CED=EBD, BDE; and is therefore alſo greater than either of them, conſequently CAD being greater than CED, it is alſo greater than CBD; *which was to be proved.*

PROBLEM 5. TO *inſcribe in a Circle, a Triangle Equi-angular to a given Triangle.* Eucl. 4. Prop. 2. 100

LET the given Triangle be FED, which muſt be inſcribed in the Circle BAC, draw the Tangent GH,‡ and make the Angle GAB equal to FDE, and CAH equal to DFE,§ compleat the Triangle ABC by drawing the Line BC. Since the Angle BAG, whoſe meaſure is half the Arc BA,⸸ (which is alſo the meaſure of the Angle BCA) is equal to FDE, then BCA=FDE. By the ſame Reaſon CAH, (equal to DFE,) having for its meaſure half the Arc AC, (the meaſure of CBA,) it muſt be, that CBA and DFE are equal; hence the two Triangles ABC and EFD, having two Angles equal, they are entirely Equi-angular,‖ having therefore inſcribed the Triangle ABC, that which was propoſed will be done.

PROBLEM 6. TO *deſcribe about a Circle, a Triangle Equi-angular to a given Triangle, or to circumſcribe a Circle with a Triangle Equi-angular to a given Triangle.* Eucl. 4. Prop. 3. 101

THE given Triangle is KLM, and the Circle is X, having drawn the Radius AD, make on one Side thereof the Angle BAD=NLK,‡ and on the other DAC=KMH; then having drawn through the three Points B, D, C, the Tangents EF, FG, and GE;¶ I ſay, that the Triangle EFG, is Equi-angular to LKM. The ſix Angles of the two Triangles BAD and BED, make four right Angles,† AB and AD being Perpendicular to each other, EBD+ABD make a right Angle, alſo EDB+BDA make a right Angle, therefore BED+BAD make two right Angles; now by Conſtruction, BAD=KLN, the Angle KLN+
KLM

* *Sup. n.* 92.
† *Sup. n.* 73.
‡ *B.* 1. *n.* 112.
§ *Sup. n.* 29.
⸸ *Sup. n.* 37
‖ *Sup. n.* 80.
‡ *Sup. n.* 29.
¶ *B.* 1. *n.* 121.
† *Sup. n.* 75.

KLM makes two right Angles, therefore BED=KLM. The ſame way it is demonſtrated, that DGC=KML, and conſequently that EFG=LKM, and that the Triangles EFG, and LKM, are therefore Equi-angular.§

102 PROBLEM 7. *To inſcribe a Circle in a Triangle.* Eucl. 4. Prop. 4.

ACCORDING to what has been taught,* divide the Angles CBD, and CDB, into two equal Parts, by the Lines BA and DA, and from the Point A where theſe two Lines interſect; draw the Perpendiculars AE, AF, AG, upon the Sides of the Triangle,¶ then from A, with the extent of one of theſe Lines, deſcribe the Circle X, which will be inſcribed in the Triangle BCD; to prove which, it muſt be ſhewn that the three Lines AE, AF, AG, are equal. The two Triangles AEB, and AFB are Rectangular by Conſtruction, ſince that AE and AF were drawn perpendicularly; the Angles EBA and ABF are equal by Conſtruction; hence theſe two Triangles having two Angles equal, they are Equi-angular;* the Side AB is common, therefore they are entirely equal.† Thus AE=AF; it is demonſtrated the ſame way, that AG is equal to AF, and to AE; *which was to be proved.*

103 CORALLARY. THREE *ſuch Lines being given, as being prolonged will make a Triangle, there may be found a Point equally diſtant from each.*

THE Triangle being made, inſcribe a Circle therein, the Center thereof is the Point required.

104 THEOREM 14. TWO *Triangles having two Sides of one, equal to two Sides of the other, but which contain unequal Angles, that which hath the greateſt Angle, will alſo have the longeſt Baſe, and reciprocally, that which hath the longeſt Baſe, has the greateſt Angle.* Eucl. 1. Prop. 24 and 25.

LET the two Triangles be ABC, DEF, if AB=DE, and BC =EF, and the Angle B>E; I ſay, the Baſe AC>DF. 1. For if the Triangle ABC, be conceived as placed upon the Triangle DEF, ſo that the Points A and B coincide with D and E: As the Angle ABC was ſuppoſed greater than the Angle DEF, the Side BC, will not coincide with EF, but will be nearer EG, as here at EH, and the Line DH will be equal to AC,* then imagining the Line EF=EH as turning round upon E, the End of DE, as it approaches to the other End EG, when it is at the Point H, the Line DH which will join their other Ends, will be longer than DF,⸸ which was to be demonſtrated in the firſt place. 2. The inverſe Part hereof will be demonſtrated in like Manner, (*viz*) That the Angle DEH=ABC, is greater than the Angle DEF, the Line AC or DH being ſuppoſed longer than DF, and the Line EH being ſuppoſed nearer than EF to the Line EG, when turned round upon the Point E, as before-mentioned.‡

SECT.

---

§ *Sup. n.* 80.

* *Sup. n.* 31.

¶ *B.* 1. *n.* 46.

* *Sup. n.* 80.

† *Sup. n.* 96.

* *Sup. n.* 89.

⸸ *B.* 1. *n.* 100.

‡ *B.* 1. *n.* 100.

SECT. IV.

OF Figures having ſeveral Sides.

105 DEFINITION I. QUADRILATERALS, *or Figures of four Sides, have different Names.*

1. IF the oppoſite Sides are parallel, 'tis a *Parallelogram.* 2. If the four Sides are equal, and all the Angles right ones, 'tis a *Square*, as A. 3. If the four Sides are equal, and alſo the oppoſite Angles equal, but not right ones, 'tis a *Rhombus*, or *Lozenge*, as B. 4. If the Sides are not all equal, but all the Angles are right ones; 'tis a *long Square, Oblong, rectangled Parallelogram*, or ſimply a *Rectangle*, as C. 5. If only the oppoſite Sides are equal, and the oppoſite Angles alſo equal, but not right ones, the Figure is called a *Rhomboides*, as D. 6. All other Quadrilaterals, whoſe oppoſite Sides are neither parallel nor equal, are called *Trapezias*, as E.

106 DEF. 2. *A Figure is called Regular, when all its Sides and all its Angles are equal.*

107 DEF. 3. *A Figure of ſeveral Sides is in general called a* Polygon, *it takes the Name which is proper to it, from the Number of its Sides, or from the Number of the Angles which its Sides contain: Thus a Figure of five Sides is called a* Pentagon; *of ſix, a* Hexagon; *of ſeven, a* Heptagon; *of eight, an* Octagon; *of nine, a* Enneagon; *of ten, a* Decagon; *of eleven, a* Endecagon; *of twelve, a* Dodecagon, &c.

108 DEF. 4. *A regular Figure is ſaid to be inſcribed in a Circle, when all its angular Points are in the Circumference of the Circle; but when all its Sides touch the Circumference of the Circle, it is ſaid to circumſcribe the Circle.*

109 DEF. 5. TWO *Lines being drawn from the Ends of one of the Sides of a regular Figure, to the Center of the Circle in which it is inſcribed, (or which it circumſcribes,) make an Angle, which is called an Angle at the Center, and the Angles which the Sides make, taken two and two, are called Angles of the Figure.*

110 LEMMA 1. OBLIQUE *Lines which make equal Angles between the ſame Parallels, are equal.*

LET there be drawn the Perpendiculars AC and DF between the Parallels Z and X, they are equal,* but the Angles ACB and DFE being right ones, and the Angles ABC and DEF equal by the Hypotheſis, the two Triangles ABC and DEF are then Equiangular, † wherefore AC is equal to DF, they are all equal; ‡ conſequently AB=DE. Q. E. D.

111 LEMMA 2. LINES *which make the ſame Angles, upon one and the ſame Line, are parallel.*

---

* *B.* 1. *n.* 65. † *Sup. n.* 80. ‡ *Sup. n.* 96.

THE Lines AB and DE make upon Z and X, the ſame Angles, hence ABC=DEF, conſequently AB and DE ought to be parallel.*

112 LEMMA 3. TWO *Lines which join two equal and parallel Lines, are equal.* Eucl. 1. Prop. 33.

IF AD and BE, join the two equal parallel Lines AB and DE, I ſay, that they are alſo equal, the Perpendiculars AC and DF are parallel,† AB and DE being parallel, the Angles ABC and DEF are equal,‡ now ACB and DEF are right Angles; therefore the two Triangles ABC and DEF, being Equi-angular § and having one Side equal, ſince that AB=DE, they are entirely equal;* therefore BC=EF; then CF=BE: Now AD and CF which are between the Parallels AC and DF, are equal,† therefore BE=CF=AD; hence BE=AD.

113 THEOREM 1. THE *four Angles of a Quadrilateral, are equal to four right Angles.*

LET the Quadrilateral be ABCD, it is required to prove, that its four Angles are equal to four right Angles: the Line AC being drawn from any one Angle, to the oppoſite Angle, it will divide the Figure into the two Triangles ABC and ACD, the three Angles of each Triangle make two right Angles.‡ Hence all the Angles of ABCD, are equal to four right Angles.

114 THEOREM 2. THE *oppoſite Angles of a Quadrilateral inſcribed in a Circle, make two right Angles.* Eucl. 3. Prop. 22.

THE Quadrilateral ABCD is inſcribed in a Circle; the two oppoſite Angles ACD and ABD, are each meaſured by half the Arc upon which they are placed; § now they all together being placed upon the whole Circumference, are therefore meaſured by half the whole Circumference, are therefore equal to two right Angles.‖ It is the ſame of the other two Angles BAC and BDC.

115 CORALLARY. IF *the oppoſite Angles of a Quadrilateral do not make two right Angles, it cannot be inſcribed in a Circle.*

FOR if it cou'd, the oppoſite Angles of the Quadrilateral would be equal to two right ones by the Theorem, but by the Suppoſition they are not; which is impoſſible.

116 THEOREM 3. IF *the oppoſite Sides of a Quadrilateral are equal, they are parallel.*

AB=CD, and BC=AD, the Side BD is common; then the two Triangles ABD and BCD being equal, they are Equi-angular:* Therefore the Angle ABD=BDC, wherefore BC is parallel to DC;† ſo likewiſe BC is parallel to AD, by reaſon the Angle ADB is equal to DBC.

117 THEOREM 4. IF *two of the oppoſite Sides of a Quadrilateral are equal and parallel, the other two are alſo equal and parallel.* Eucl. 1. Prop. 23.

THEOREM

---

* *Sup. n.* 28.

† *B.* 1. *n.* 68.

‡ *Sup. n.* 27.

§ *Sup. n.* 80.

* *Sup. n.* 95.

† *B.* 1. *n.* 65.

‡ *Sup. n.* 75.

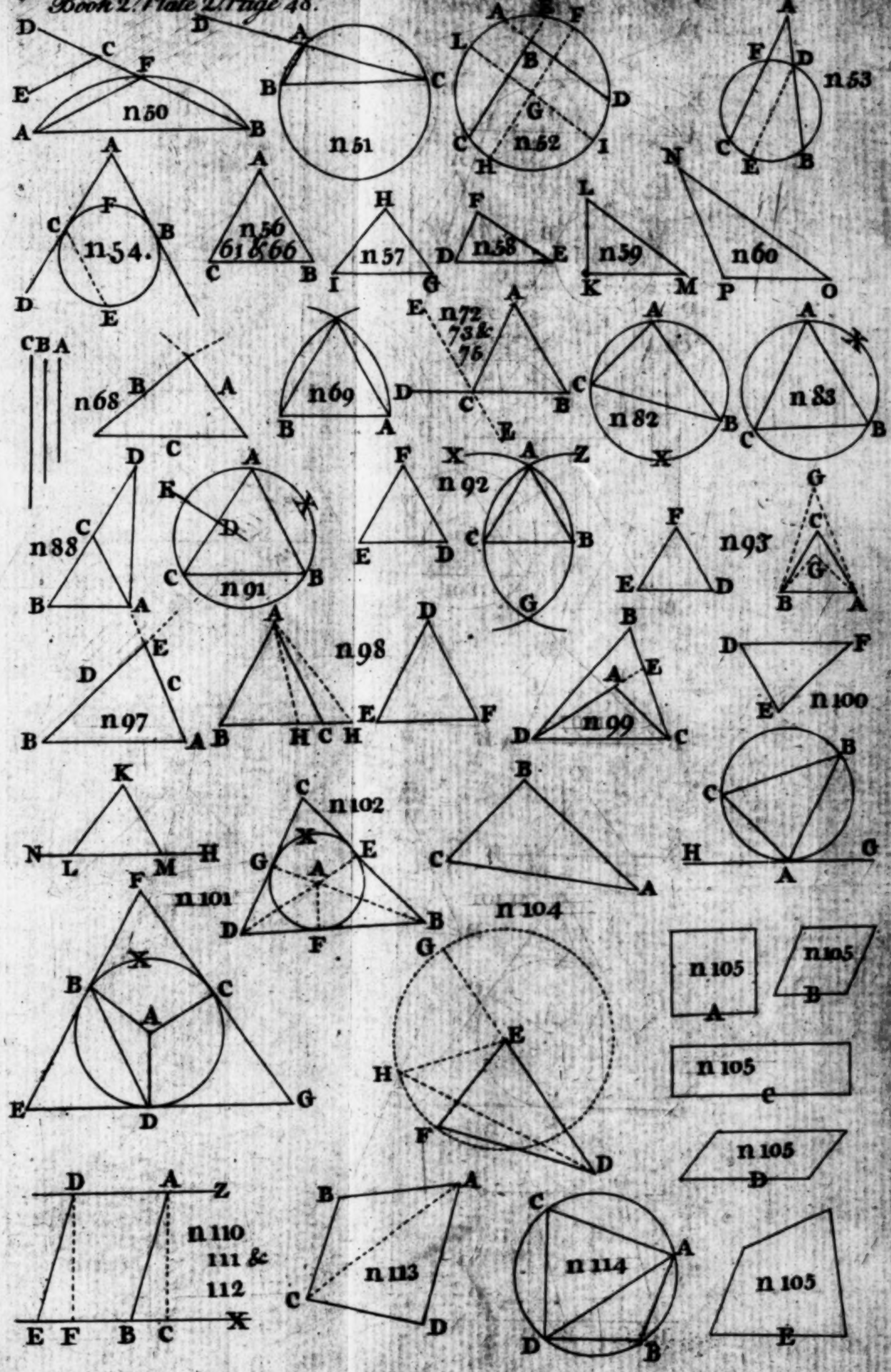
§ *Sup. n.* 39.

‖ *Sup. n.* 47.

* *Sup. n.* 94.

† *Sup. n.* 26.

Book 2. Plate 2. Page 48.



Theorem 5. If *the four Angles of a Quadrilateral are right ones, it is a Parallelogram.* 118

For AB and CD by the Hypotheſis are Perpendicular upon AC; therefore they are parallel,* AC and BD are alſo Perpendicular upon DC; conſequently by the ſame reaſon, they are parallel.

Theorem 6. The *oppoſite Angles of a Parallelogram are equal, and thoſe which are contiguous are equal to two right ones.* 119

1. The Angle $FDA=DCB$,† and $FDA=DAB$,‡ wherefore ſince two Angles equal to a third, are equal, $DAB=BCD$: Now $FDA+ADC$ makes two right Angles;§ then BCD equal to ADF, makes with ADC two right Angles. 2. So likewiſe it may be demonſtrated, that the two oppoſite Angles ABC and ADC are equal, and that $BAD+ABC$ is equal to two right Angles. Q. E. D.

Problem 1. To *make a Parallelogram having one Angle, and the two containing Sides given.* 120

The given Sides are Z and X, the given Angle is K: Z and X muſt be joined, ſo as to make an Angle equal to K,‖ and then draw two Lines parallel to Z and to X.¶

Corallary. Therefore *to make a Square having one Side given, there is no more required than to join two Lines equal to that which is given, ſo as they may make a right Angle.* Eucl. 1. Prop. 46. 121

Theorem 7. Every *Polygon, or Figure of ſeveral Sides, may be divided into as many Triangles as it has Sides, leſs two.* 122

In the Polygonal Figure X, having drawn from one of its Angles as A, right Lines to all the other Angles, it will be thereby divided into ſeveral Triangles, whoſe Baſes are the Sides of the Polygon, except two which are at the two Sides of A, of which AB and AF are one of the Sides; hence there is as many Triangles, leſs two, as there are Sides, 'tis the ſame in all Polygons; wherefore it may be abſolutely ſaid, that a Polygon may be reduced to as many Triangles, as it hath Sides, leſs two.

Corallary. Therefore *all the Angles of a Figure which hath ſeveral Sides, are equal to twice ſo many right Angles, leſs four, as the Figure hath Sides.* 123

That is, all the Angles of X, which hath ſix Sides, are equal to eight right Angles; for the Polygon is divided into as many Triangles as it hath Sides, leſs two, *that is,* into four. Then ſince the Angles of each Triangle are equal to two right ones, all the Angles of the four Triangles together, are equal to eight right Angles: But they all together compoſe the ſix Angles of the Figure, therefore they are equal to eight right ones; *that is,* to twelve right Angles, leſs four, it may therefore be ſaid, that all the Angles of a Chiliogon, *that is,* a Figure of a thouſand Sides, are equal to 1996 right Angles, this may be eaſily conceived, although it is impoſſible to conceive a Chiliogon exactly.

---

* *B.* 1. *n.* 68. † *Sup. n.* 27. ‡ *Sup. n.* 25.
§ *Sup. n.* 17. ‖ *Sup. n.* 29. ¶ *B.* 1. *n.* 72.

ADVERTISEMENT.

By regular Figures, we mean thoſe whoſe Sides and Angles are all equal,[*] and which may conſequently be inſcribed in a Circle, the Angle EAF, is called the Angle at the Center, becauſe it is formed at the Center of the Circle by the Radii AE and AF, drawn to the Ends of the Sides of the Polygon, and the Angles ſimilar to GEF, formed by the meeting of the Sides of the Polygon, are called Angles of the Polygon, and as it has been demonſtrated,[†] all the Angles formed at the Point A, which is here taken for the Center of the Polygon, are together equal to four right Angles. When the bigneſs of each Angle at the Center is required, as here in the Example of the Hexagon, the Sum of four Angles which is 360 Degrees, muſt be divided by ſix, the Number of the Sides of the Polygon, the Quotient which is 60, is the content of the Angle at the Center: The Angle at the Center being known, that at the Polygon will be likewiſe known; for as the three Angles of the Iſoceles Triangle EAF, are together equal to two right ones;[‡] or 180 Degrees, ſubtracting therefore the Angle at the Center, the remainder is the Sum of the two equal Angles at the Baſe, or Side of the Polygon; which are equal to GEF only, the Angle of the ſaid Polygon, hence that of the Hexagon is 120 Degrees: Which might be found alſo, by what has been demonſtrated;[§] ſince that the Angle GEF, is meaſured by half the Arc of the Circle GHF, upon which it ſtands, therefore ſubtracting the Part GF, here ſuppoſed 120, from 360 Degrees, which make the whole Circle, there will remain 240, half of which, is the content of GEF, or Angle of the Polygon.

QUESTION.

What *Polygons are they, which may be joined by their angular Points, without leaving a Vacuity?*

They muſt be ſuch, that the Angles thereof, which are about the common Point, make exactly four right Angles,[¶] hence, there is only Triangles, Squares and Hexagons that can do it; the four Angles of four Squares, which have one Point common, make four right Angles, the three Angles of three Hexagons, having one Point common, being each 120 Degrees, they together make 360, the Sum of four right Angles. Six Equilateral Triangles, being placed about a Point, their ſix Angles touch, and make four right Angles.

When the Angle at the Center of a regular Figure is known, it may be inſcribed in a Circle, by drawing from the Center two Radius's, making ſuch an Angle, as the Angle at the Center of the Figure ought to be. For if it is a Figure of ten Sides, which makes an Angle of thirty-ſix Degrees, which is the tenth Part of 360, the Chord of that Angle will be one of the Sides of that Figure.

---

[*] *Sup. n.* 106.

[†] *Sup. n.* 19.

[‡] *Sup. n.* 75.

[§] *Sup. n.* 39.

[¶] *Sup. n.* 19.

To Circumſcribe a Polygon, or regular Figure about a Circle, it muſt be firſt inſcribed therein and the Radii be produced, then having divided one of the inſcrib'd Sides of the Polygon into two equal parts, as EF, by drawing the Radius AB thro' the middle thereof, and having drawn at the PointB, a tangent between AC and AD it will be one of the Sides of the circumſcribing Polygon, then all the Radii of the inſcribed Polygon, muſt be prolonged equal to AC and AD, thro' the Ends of which, having drawn right Lines, you will have the Figure required; which is evident, but the Angle at the Center of every regular Figure, cannot be deſcribed with the Rule and Compaſs only, as ſhall be made appear, but only mechanically, by means of a Semi circle divided into Degrees called a *Protractor*.

PROBLEM. 2. To *inſcribe a Square in the Circle X.* Eucl. 4. 124
Prop. 6.

THE Circle given is X, to inſcribe therein a Square, thro' the Center muſt be drawn a Line ſuch as AC, which is a Diameter, thro' the middle thereof draw the Perpendicular BD, the four Points A. B. C. D. are equally diſtant, * then having drawn right Lines thro' theſe four Points, there will be a Square in the Circle X. 1. This Figure hath four equal Sides. 2. All the Angles of A, B, C, D, are right ones. † For each of them ſtands upon the Semi-circumference.

PROBLEM 3. To *inſcribe a Circle in a Square.* Eucl. 4. 125
Prop. 8.

THE Square FGHI is given to inſcribe a Circle therein, having divided each of the four Sides of the Square into two equal Parts, and drawn AC and BD, if a Circle be deſcribed from the Point E where they interſect, with the extent of AE, it will be inſcribed in the Square. For the four Lines AE, BE, CE, DE are equal, hence, the Circle will paſs thro' the Points A, B, C, D.

PROBLEM 4. To *circumſcribe a Square about a Circle.* Eucl. 4. 126
Prop. 7.

THE Circle ABCD (*fig. preceed,*) is given to be circumſcribed by a Square, there muſt be drawn two Diameters croſſing at right Angles, then having drawn thro' the four Ends of theſe Diameters, four Tangents to the Circle, ‡ they will (*as is evident*) form the Square required.

PROBLEM. 5. To *deſcribe a Circle about a Square.* Eucl. 4. 127
Prop. 9.

THE Square ABCD is given to be circumſcribed by a Circle. Having drawn diagonals, *that is*, right Lines from one Angle to the other oppoſite thereto, and taken half one of the Diagonals for Radius, A Circle deſcribed therewith will be that which is required.

---

* *b. 1. n. 42.* † *Sup. n. 44.* ‡ *b. 1. n. 112.*

SECT. V.

Of the Area of Surfaces.

Theorem. I.

128 IN *every Parallelogram, the oppoſite Sides and Angles are equal one to another, and the Diagonal divides it into two equal parts* Eucl, 1. Prop. 34.

Let X be a Parallelogram whoſe Diagonal is AB: It muſt be proved, that the Angle ADB is equal to ACB, and that the Sides AC and BD are equal: The two oppoſite alternate Angles ABD and BAC are equal, * by the ſame reaſon BAD=ABC, the Side AB is common; hence the two Triangles ABD, BAC are entirely equal; † then the Angle D is equal to the Angle C, and the Angle A to the Angle B, ſeeing they are compoſed of equal Angles, and AD=CB, as alſo AC=BD, and AB equally divides the Parallelogram ABCD.

129 Theorem 2. Parallelograms *which are between the ſame Parallels, and upon the ſame or equal Baſes, are equal.* Eucl. 1. Prop. 35 and 36.

If the Parallelograms ABCD, EFCB are between the ſame Parallels Z and X, and upon the ſame Baſe BC, or another equal thereto, I ſay, that they are equal. If the Baſe of the Parallelogram BCEF, is different from that of ABCD, upon BC let there be made the Parallelogram EFCB, ‡ ſimilar and equal to the given one EFCB, it is to be proved, that the laſt Parallelogram EFCB, is equal to the Parallelogram ABCD. The Sides AB, CD of the Parallelogram ABCD, are Parallel by ſuppoſition, they are alſo equal, § ſo likewiſe are the Sides EB, FC. But if ED is added to AD and EF, which are ſuppoſed equal, the wholes AE, DF are equal, hence the two Triangles EAB, FDC having their three Sides equal, they are entirely equal; ‖ and ſubtracting from theſe two equal Triangles, their common part DGE, the Trapezia's ABGD and CGEF are equal; then adding to each the ſame Quantity BGC, which makes the Parallelograms ABCD and BCEF, the two Figures are equal, *which was to be proved.* Obſerve that if the Point E, be between D and A, inſtead of adding DE it muſt be ſubtracted, in order to conclude the equality of the Lines AE and DF, and afterwards that of the two Triangles, EAB, FDC, and conſequently that of the two Parallelograms, by means of the common Trapezia, which is to be added.

---

* *Sup. n.* 25. † *Sup. n.* 96. ‡ *Sup. n.* 120.
§ *Sup. n.* 128. ‖ *Sup. n.* 92.

130 CORALLARY. 1. THEREFORE *in meaſuring the Superficies of a Parallelogram, regard is had to only the Baſe, and the Perpendicular Height.*

FOR (*fig. preceed.*) the Parallelogram BCEF, is equal to a rectangled Parallelogram whoſe Baſe is BC, and whoſe Sides being Perpendicular, are equal to its Height, thus the rectangled Parallelogram ABCD, is the Meaſure of all the Parallelograms whoſe Baſes are equal to BC, and which have the ſame height, or are between the ſame Parallels Z and X; there can be but one right angled Parallelogram between X and Z upon the Baſe BC, and there may be an infinite number of Oblique angled Parallelograms upon the ſame Baſe, and between the ſame Parallels X and Z.

131 CORALLARY. 2. THEREFORE *in meaſuring the extent of an Oblique angled Parallelogram, no regard is had to its Circumference.*

FOR altho' the Sides BE and CF were of a Million, or an infinite number of Miles, which may be conceived ſuppoſing the Lines X and Z to be prolonged *ad inf.* the Parallelogram whoſe Circumference is infinite, is no greater than ABCD whoſe Circumference is finite.

132 THEOR. 3. IF *two Lines be drawn Parallel to the Sides of a Parallelogram, and thro' any Point whatſoever of its Diagonal; they will divide the Parallelograms into four Parts, of which thoſe two, which the Diagonal does not paſs thro', and which are called complements, are equal one to another.* Eucl. 1. Prop. 43.

ABC=ADC, and AGF=AKF, and FEC=FHC * Subtracting from the equal Triangles ADC and ABC, the equal Quantities AKF and FHC on one Side, and AGF and FEC on the other, the remainders FKDH and BEFG, are equal. Q. E. D.

133 THEOR. 4. PARALLELOGRAMS *are double the Triangles of the ſame Baſe and Height.* Eucl. 1. Prop. 41.

THE Triangle ECD hath the ſame Baſe as the Parallelogram ABCD, and they being both between the Parallels X and Z, have the ſame Altitude; draw DF Parallel to CE, making the Parallelogram DCEF, which is equal to ABCD. † Now CED is equal to EDF, ‡ therefore DCEF or its equal ABCD, is the double of CED.

134 CORALLARY. 1. THEREFORE *Triangles of the ſame or equal Baſes, and having the ſame height, are equal.* Eucl. 1. Prop. 37 and 38.

BY reaſon they are each the half of a Parallelogram of the ſame Baſe and height.

135 CORALLARY. 2. THEREFORE *in meaſuring a Triangle, regard is had to only its Baſe and height.*

---

* *Sup, n.* 128. † *Sup. n.* 129. ‡ *Sup. n.* 128.

ADVERTISEMENT.

To meaſure the Triangle ABC, there muſt from the Vertex be let fall a Perpendicular AD, which is the Altitude thereof, and BC muſt be multiplied by half AD, or AD by half BC, or BC by AD, and that Product divided by 2, all which amount to the ſame. If the Triangle were obtuſe-angled, as MNP, then in order to obtain the height thereof, produce the Side MP, upon which let fall a Perpendicular NQ, from the Vertex N.

But *if the three Sides of the Triangle are known, its Content may be found without drawing a Perpendicular, thus.* Add the three Sides together into one Sum, and from half that Sum, Subtract each Side ſeverally, ſo will there be three Remainders, then multiply the half Sum of the three Sides, by one of thoſe Remainders, and that product by another, and this laſt product by the third or laſt Remainder, then will the Square-Root of this laſt Product, be the Area, or Content of the Triangle required.

136 Theor. 6. Equal *Triangles having the ſame or equal Baſes, and placed on the ſame Side, are between the ſame Parallels,* Eucl. 1. Prop. 39 and 40.

If they are equal, they have the ſame Altitude, * and Conſequently the Perpendiculars let fall from the Vertex of each, to its correſponding Baſe, are equal. Therefore if a Line be drawn thro' the Vertexes of thoſe two Triangles, it will be parallel to the Baſes thereof. †

ADVERTISEMENT.

137 If there are two Triangles whoſe Superficies are Equal, that which hath the moſt Sides and Angles equal one to another, hath the leaſt Circumference, a right angled Triangle, hath a greater Circumference than an Equilateral one which is equal thereto; for let ABC be a right angled Triangle, and ABD an Equilateral one having the ſame height, they are Equal, ‡ Prolong AC to E, making $AC=CE$, the Circumference ABC is $AB+BC+CE$; Produce BD to E, each Angle of ABD is 60 Degrees, and EAB being Ninety, the Angles AEB and EAD are each 30, therefore EDA is an Iſoceles, and $DE=AD$, then $AB+BE$ is the Circumference of ABD; now BE is ſhorter than $BC+CE$; therefore the Circumference of ABD is ſhorter than that of ABC, The more regular and uniform a figure is in all its parts, the greater is its contained Surface, and the leſſer its Circumference. As will appear farther on.

138 Problem. 1. To *deſcribe a Parallelogram having one Angle thereof given, equal to a Triangle given.* Eucl. 1. Prop. 42.

---

* *Sup. n.* 134. † *b.* 1. *n.* 64. ‡ *Sup. n.* 134.

The given Triangle is ABC, and the given Angle D; required to make a Parallelogram having an Angle equal to D, and equal to the Triangle ABC. 1. Thro' B the Vertex of the Triangle, draw BF parallel to its Baſe AC. * 2. Upon E the middle of that Baſe, draw GE making with AC an Angle equal to D. † 3. Compleat the Parallelogram CEGF, ‡ which will be that which was required; for it is equal to the Triangle ABC, § and hath the Angle GEC equal to the given Angle D.

Problem. 2. To *deſcribe upon a given right Line, a Parallelogram equal to a given Triangle, having one Angle equal to a rectilinear Angle given.* Eucl. 1, Prop. 44. 139

The given Line is A, the given Angle is C, required to make a Parallelogram equal to the Triangle B. 1. By the preceeding Problem, make the Parallelogram, DEFG equal to the Triangle B, having an Angle equal to C. 2. Prolong GF and DE, ſo that $FH=A$, and $EI=FH$; compleat the Parallelogram EFHI, and produce the Line IF till it meet the produced Part of DG, and compleat the Parallelogram DKLI, in which is FHLM equal to DEFG; ‖ hence FHLM is the Parallelogram required, equal to the Triangle B (to which DEFG was made equal,) having an Angle equal to the given Angle C, and being made upon FH equal to the given Line A.

Problem 3. To *deſcribe a Parallelogram equal to a given right lined Figure, and having one Angle equal to a given right lined Angle.* Eucl. 1. Prop. 45. 140

The given right lined Figure is ABCD, the given Angle E. 1. Reduce the Figure ABCD into two Triangles, by the Line AC. 2. Draw the Parallelogram GI equal to the Triangle ABC, having the Angle G equal to the given Angle E. * 3. Upon HI draw alſo the Parallelogram IL equal to the Triangle ACD, having the Angle IHL equal to the ſaid Angle E, † this being done, $GK=GI+IL$; therefore $GK=ABCD$ is the Parallelogram required. But if the Figure had been compoſed of more than two Triangles, the ſame thing would have been done upon the Line KL for the third Triangle, as was done upon HI for the Second; and ſo likewiſe for the reſt.

Definition. The *Side oppoſite to the right Angle of a rectangled Triangle, is called Hypothenuſe.* 141

Theorem 6. In *every rectangled Triangle the Square of the Hypothenuſe, or ſide which ſubtends the right Angle, is equal to the Squares of both the other Sides.* Eucl. 1. Prop. 47. 142

Let the rectangled Triangle be CAB, if upon the Hypothenuſe CB be made the Square CD, (*a Parallelogram or Square is denoted by two diagonal Letters*) and upon the other two Sides CA and AB, be made the Squares CK, BG: I ſay, that $CD=CK+BG$. From the Vertex A, let there be drawn the Line AL

---

* *b.* 1. *n.* 72. † *Sup. n.* 29. ‡ *b.* 1. *n.* 72.
§ *Sup. n.* 133. ‖ *Sup. n.* 132.
* *Sup. n.* 138. † *Sup. n.* 139.

Perpendicular upon the Hypothenuſe BC, which will divide the Square CD into the two Parallelograms IE and ID, the firſt of which is equal to the Square CK, and the ſecond to the Square BG; which is thus demonſtrated. From the Point A to the Point E, let there be drawn the Line AE, and from B to the Point H the Line BH, theſe two Lines will form the two equal Triangles ACE, BCH,* for the Side CE=CB, and CA=HC, and the Angles ACE and HCB which thoſe Sides contain are equal, having each a right Angle, and the Angle ACB common; but the Triangle ACE is half the Parallelogram IE,† becauſe they have both the ſame Baſe and are both between the ſame Parallels, the Triangle HCB, is likewiſe for the ſame Reaſon, half the Square CK, now theſe Triangles are equal; therefore the Parallelograms of which they are the Halves, are alſo equal.

What has been ſaid of the Parallelogram IE, and of the Square CK, is to be underſtood of the Parallelogram ID, and of the Square BG, therefore, &c.

143 Problem 4. To *deſcribe a Square equal to two, or to ſeveral other Squares.*

Join the two Baſes AB, and AC, of the two Squares propoſed, ſo that they make a right Angle, the Baſe CB of that Angle will be the Side of a Square equal to both the Squares, according to the preceeding Theorem. If a Square be required equal to three given Squares, and that BD is the Side of the third, join BC and BD, ſo that they make a right Angle, then the Square of CD is equal to the Squares of BC and BD, conſequently to the Squares of AC, AB, BD; and by this means a Square may be formed equal to as many Squares you pleaſe.

144 Theorem 7. If *the Square of one of the Sides of a Triangle is equal to the Squares of the other two Sides, the Angle contained between thoſe two Sides is a right one.* Eucl. 1. Prop. 48.

I ſuppoſe the Square of the Side BD of the Triangle ABD, to be equal to the Squares of the other two Sides AB, AD, if ſo, I ſay, that the Angle BAD contained between the two Sides AB, AD, is a right one. Draw AC Perpendicular upon AD and make it equal to AB, then the Squares of AD and AC are equal to that of DC,* now by the Hypotheſis, theſe two Squares are equal to that of BD; the Squares of BD and DC are therefore equal. Hence CD and BD are equal; then the two Triangles ABD and ADC are entirely equal, and conſequently Equi-angular,† BAD is therefore a right Angle as well as CAD, it is here ſuppoſed that equal Squares have equal Sides, which is evident.

145 Theorem 8. *A Triangle is equal to two or more Triangles having the ſame Height, if its Baſe be equal to all their Baſes taken together.*

The Triangle ABC is equal to the Triangles ABE, EAD and DAC, which are its Parts. Now ACD=CFD, and DAE=DGE, and EAB=EHB,‡ therefore CAB is equal to two or more Triangles, &c. *which was to be proved.*

---

* *Sup. n.* 98. † *Sup. n.* 133. * *Sup. n.* 142.

† *Sup. n.* 92. ‡ *Sup. n.* 134.

DEFINITION. THE *Part of Radius which is Perpendicular between the Side of a Polygon, and the Center of the Circle, is called* Apotome. 146

THEOREM 9. THE *ſuperficial Content of a regular Polygon, is equal to a Triangle whoſe Baſe is equal to the Circumference of the Polygon, and its Height equal to the Apotome thereof.* 147

HAVING drawn Lines from the Center A to each Angle, it will be reduced into as many Triangles as it hath Sides, all equal to the Triangle ABC whoſe Baſe is BC, and its Height the Apotome AD, now a Triangle whoſe Height is AD, and its Baſe the Circumference of the Polygon, is equal to all theſe Triangles whoſe Height is DA; and the Baſes taken together, make the Circumference of the Polygon;* *which was to be proved.*

EVIDENT Propoſitions concerning Polygons.

PROP. 1. *A Polygon is greater than its inſcribed Circle.* 148

PROP. 2. *A Polygon is leſſer than its circumſcribing Circle.* 149

PROP. 3. *A Circle may be eſteemed as a Polygon having an infinite Number of Sides.* 150

FOR if a Polygon of a ſmall Diameter hath a Million of Sides, it is evident, that the difference between it and a Circle is not preceivable, if I ſay, there be imagined in a Circle as many Sides are there are viſible Points, it will be a Polygon and a Circle at the ſame Time.

THEOREM 10. OF *two regular Polygons which circumſcribe a Circle, that which hath moſt Sides, hath the leaſt Circumference and the leaſt Surface.* 151

X is a Polygon which circumſcribes a Circle, I divide its Sides in order to make another Polygon having more Sides, by drawing Tangents which are without the Circle;† hence the Polygon will be greater than the Circle,‡ let us conſider the ſame Part of the two Polygons, *that is*, the Part which circumſcribes the ſame Part of the Circle, for Example EFD, ſince BA+AC is longer than BC, then BD+BA+AC+CF is longer than DB+BC+CF: Hence the Circumference of that which hath feweſt Sides is greateſt. The Figure EFCABD exceeds EFCBD by the bigneſs of the Triangle ABC; therefore the Surface of that which hath moſt Sides is leaſt.

CORALLARY. THEREFORE *ſince the more Sides a circumſcribing Polygon hath, the leſſer it is, (always remaining greater than the Circle which it circumſcribes) it follows, that the more Sides a Polygon has, the nearer its Circumference and Surface approach to the Circumference and Surface of the Circle which it circumſcribes, and that therefore a circumſcribing Polygon having an infinite Number of Sides, is the ſame as a Circle.* 152

THEOREM 11. IF *there are two regular Polygons inſcribed in the ſame Circle, that which hath the moſt Sides, hath the greateſt Circumference, and the greateſt Surface.* 153

---

* *Sup. n.* 143. † *b.* 1. *n.* 106. ‡ *Sup. n.* 146

Two Polygons being inſcribed in the Circle X; let us conſider the part ABDC, and the correſponding Parts of the two inſcribed Polygons. 1. BD+DC is longer than BC, wherefore the Circumference of the Polygon having moſt Sides, appears already to be the longeſt. 2. The Figure ABCD exceeds ABC, by the bigneſs of the Triangle BDC, therefore the Polygon which hath moſt Sides hath the greateſt Surface, which was to be proved.

154 CORALLARY. THEREFORE *ſince two or more Polygons inſcribed in the ſame Circle, that which hath moſt Sides is greateſt, ſo long as it remains leſſer than the Circle,* * *it from hence follows, that the more Sides an inſcribed Polygon hath, the nearer it approaches to the Circumference and Surface of the Circle, and that therefore an inſcribed Polygon, the Number of whoſe Sides is infinite, is the ſame as the Circle.*

155 THEOREM 12. OF *two regular Polygons inſcribed in one and the ſame or equal Circles, that which hath moſt Sides hath the longeſt Apotome.*

EC is the Chord of the Polygon which hath moſt Sides, CI the Chord of that which hath the feweſt, hence CE being the Chord of a leſſer Arc, is therefore ſhorter, and conſequently farther from the Center A than CI,† then the Perpendicular AD which is the Apotome of the Polygon whoſe Chord is CE, is longer than the Apotome of the Polygon whoſe Chord is CI, *which was to be proved.*

156 THEOREM 13. THE *Surface of a Circle is equal to a Triangle whoſe Height is the Radius, and its Baſe the Circumference of the Circle.*

ACCORDING to the two preceeding Theorems and their Corallaries, a Circle may be ſuppoſed as equal to a circumſcribing or inſcribed Polygon, the Sides whereof are infinite. The Surface of ſuch a Polygon, is equal to a Triangle whoſe Baſe is the Circumference thereof, (*which is the ſame thing as the Circumference of the Circle,*) and whoſe Height is the Apotome, or Perpendicular drawn from the Center of the Polygon upon one of the Sides thereof, the Sides of which being infinite, that Perpendicular will be the ſame as the Radius of the inſcribed Circle; taking therefore the Circle for a ſimilar Polygon, its Surface is equal to a Triangle, whoſe Baſe is equal to its Circumference, and the Height equal to the Radius thereof.

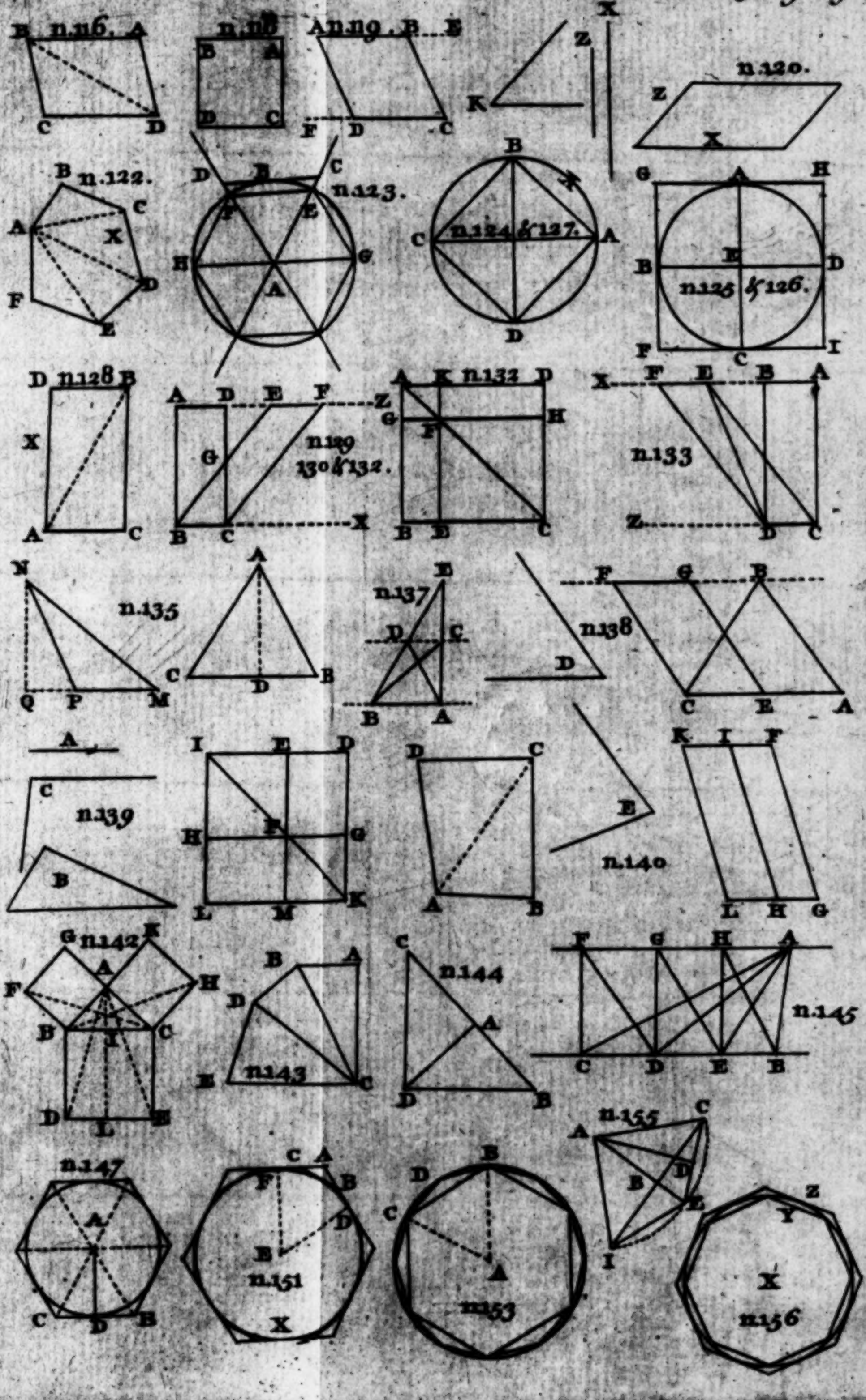
THE Radius of a Circle may be eaſily meaſured, but how to meaſure the Circumference remains yet unknown, which prevents our finding the Quadrature of the Circle, that is, of making a Square equal to the Surface contained in a Circle, for if the Circumference were known, it would be eaſy to form a Triangle equal to the Circle, by making its Baſe equal to the Circumference, and its Perpendicular equal to the Radius, the Triangle (as will hereafter appear) may be eaſily transformed into a Square equal to it, and conſequently equal to the ſaid Circle, but the Circumference of the Circle can be no otherways

---

* *Sup. n.* 147.

† *B.* 1. *n.* 96.

Book 2. Plate. 3. Page 59



ways expreſſed, than by aſſigning two Lines one greater, and the other leſs than the Circumference, which differ from one another, by only a certain Quantity leſs than the whole Quantity ſignified, which is thus demonſtrated.

LET the given Circle be X, and the ſuppoſed Polygon Z, which circumſcribes the Circle, and another which I call Y is inſcribed therein, let the given Quantity, which is the Difference between the Surface of the Circle to any other Surface, be T. Encreaſe or diminiſh the Number of the Sides of the Polygons Z and Y, till their Difference is leſs than the Quantity T, which is eaſily done, for by augmenting the Sides of each, the Bigneſs of Y is encreaſed,* and that of Z diminiſhed;† hence each of them approach nearer to the Circumference of the Circle; the Difference between Z and Y is greater than that of Z to the Circle X, ſince X is greater than Y, then the Difference between the Surfaces Z and Y, being leſſer than the Quantity T, there is found a Surface which differs from that of the Circle, by a Quantity leſſer than what was propoſed, that is, if it were propoſed to find a Surface differing from that of a given Circle, by only the hundred thouſandth Part of a Line, there may be found another that differs yet leſs.

---

* *Sup. n.* 152. † *Sup. n.* 149 and 150.

* * *

BOOK III.

BOOK. III.

THE Properties belonging to Quantity in general, applied to Lines, Planes and Solids.

SECT. I.

THE four Operations of Arithmetic, ADDITION, SUBTRACTION, MULTIPLICATION, and DIVISION, upon Lines, Planes and Solids.

ADDITION.

1 THE firſt and ſimpleſt Property of Quantity in general, and conſequently of a Line, a Plane, and a Solid, is that it may be either augmented or diminiſhed, one Line may be added to another, by joining or prolonging them, ſo likewiſe a Plane to a Plane, by continuing or putting their Sides together, when Quantities are expreſſed by Letters, whether they be Lines, Planes, or Solids, the Sign of Addition is +, thus $a+b$ ſignifies that the Quantities which are called a and b, (*whether they be Numbers or Lines*) are added together. When the ſame Letter is repeated, or taken ſeveral Times, it is written only once, but there is a Figure put before it, which ſhews how often it is added to itſelf; 3 b ſignifies that b is added three Times to itſelf. Thoſe Quantities which are the ſame, and have different Signs are eraſed, or cancelled. In Geometry a Line is generally denoted by two Capitals placed at the Extremities thereof, as AB, or by only a ſmall Letter which is put at the Middle:, thus a denotes the Line AB. A——— ———B

a

SUBTRACTION.

2 THE Sign of Subtraction is —, thus $a-b$ ſignifies that b is to be ſubtracted from a, and that conſequently a is greater than b. The Difference of two Quantities is the Exceſs of the greater above the leſſer, or the greater, leſs the leſſer.

Let there be two Lines AB and CD, having A———E——B taken from AB, the Line AE, equal to CD, C———D their Difference is EB, which is the Exceſs of AB above CD, or the greater Line AB, leſs the leſſer Line CD, that is, what remains of the greater, after the leſſer is deducted. If the Plane BCEF, be ſubtracted from the Plane ABCD, the Subtraction is denoted thus, ABCD—BCEF, which is the Difference of the two Planes, after the Subtraction the remainder is the Plane ADEF, the general Rule of Subtraction is this; *change all the Signs of the Quantities that are to be ſubtracted:* that is, change Plus into Minus, and Minus into Plus, the Sign + is always underſtood as before the Quantity which hath no Sign, thus to ſubtract b from a, *that is,* + b from a, the Sign + muſt be changed into —, and it muſt be wrote a—b.

MULTIPLICATION.

3 To multiply one Quantity by another, is to take one, (*it matters not which firſt*) as many Times as there are Unites in the other. To multiply a by b, is to take a as many Times as b hath Unites or Parts, which is expreſſed by joining the two Letters thus ab. When a Plane is marked by two Letters, as ab: it is ſuppoſed that a, or BC, makes a right Angle with b, or AB: In the preceeding Book we demonſtrated, that the bigneſs of a Plane does not depend upon the Length of its Sides only, but alſo upon the Nature of the Angle which is formed thereby, that a right angled Parallelogram is greater than an Oblique one:* Hence there is a determinate Meaſure, for which Reaſon it is ſuppoſed, (*as I ſhall proceed to ſhew*) that two Lines which are conceived as multiplied one by the other, make a Rectangle. When ſmall Letters are uſed, if they are joined together, it denotes that they are multiplied into each other, but it is not ſo when Capitals are uſed; thus AB does not denote that A is multiplied by B, but that AB is a Line whoſe Extremities are A and B, the Sign of Multiplication of two Lines marked with Capitals is this ×, that is, AB × BC, ſignifies that AB is multiplied by BC. If it be conceived that the Plane ABCD be taken and put upon itſelf as many Times as there are Parts in DE, it will make a Solid: ABCD×E, if AB=a. and AD=b, and DE=c, the Product will be abc, as a Plane is denoted by two Letters, ſo a Solid is denoted by three.

DIVISION.

4 THE Quantity that is propoſed to be divided, is called *Dividend:* That which is divided by, is called *Diviſor*; and that which ſhews how many Times the Diviſor is contained in the Dividend, is called *Quotient.*

To ſhew that one Quantity is divided by another, the Letters by which they are repreſented, are put under one another, and ſeparated by a little Line, the Dividing a by b, is denoted by putting the a above b in this Manner $\frac{a}{b}$, which ſhews that a is to be conceived as divided by b. When there are Letters equally contained in the Diviſor, and Dividend, they muſt be cancelled, thus to divide cd by c, the c muſt be ſuppreſſed and d remain, which will be Quotient of cd divided by c, *that is*, d ſignifies how many Times c is contained in cd.

---

* *B.* 2. *n.* 230.

5 THE ſame Operations repeated upon compound Quantities. A Quantity is called compound, when it is compoſed of two, or more Quantities, which are each expreſſed by a particular Sign, $a+b$, and $a-b$, are compound Quantities, more one Quantity leſs the ſame Quantity is nothing, $+3-3$, is equal to Cypher, therefore to render an Expreſſion elegant, thoſe Quantities which are the ſame, but have contrary Signs, are cancelled: Having for Example this Expreſſion $5b-2b$, it is reduced to this $3b$, for as was before ſaid, when a Quantity has no Sign expreſſed, the Sign + muſt be underſtood, thus $5b-2b$, is the ſame as if it was $+5b-2b$; now in this Expreſſion there is $2b$ with a contrary Sign, therefore I cancel it, and write $+3b$, or ſimply $3b$.

ADDITION *of compound Quantities.*

6 Compound Quantities are added in the ſame Manner as ſimple ones; to add $b+c$ to $f+h$, they muſt be joined by the Sign +, thus $b+c+f+h$, Letters which are the ſame, but have contrary Signs, muſt always be cancelled, thus in adding $b+d$ to $c-d$, the Sum is only $b+c$. When the ſame Letter is repeated ſeveral Times, there is put only one, with a Figure before it, ſhewing how many Times it is taken, as here to add $4f+6g$ to $3f-4g$, is wrote $7f+2g$, if this Operation had been expreſſed at length, there would have been wrote, $4f+6g+3f-4g$; now $4f+3f=7f$, and $+6g-4g=$, $+2g$. A Quantity having the Sign −, being on one ſide of the Sign =, if you would add it to the other ſide, cancel it from the Place where it ſtands, and put it on the other ſide with the Sign +, thus let $a=b-d$, to add d to the other ſide, make it $a+d=b$, this expreſſed at length is $a+d=b-d+d$, but $-d+d=0$.

SUBTRACTION *of compound Quantities.*

7 THE general Rule for this is the ſame as for ſimple Quantities, the Signs of the Quantity required to be ſubtracted muſt be changed, the Sign + ought always to be underſtood as before every Quantity which has no Sign, therefore to take $b+d$, or $+b+d$ from $c+f$, there muſt be wrote $c+f-b-d$. For 1, b muſt be joined to $c+f$ with the Sign of Subtraction. 2. 'Tis not only $+b$ that ſhould be ſubtracted, but alſo $+d$, it muſt therefore be wrote $-d$, on the contrary to take $b-d$ from $c+f$, the Signs of $+b-d$, muſt be changed, putting them $c+f-b+d$.

OBSERVE well in this Expreſſion $c+f-b-d$, it is not from b that d is ſubtracted, but b and d are ſubtracted from $c+f$. When one and the ſame Quantity is to be ſubtracted from each ſide of the Sign of Equality, it is only cancelling it in the Place where it is with the Sign +, and to put it on the other ſide with the Sign −, let $a=b+d$, to take away d from each ſide, write $a-d=b$.

MULTI-

MULTIPLICATION *of compound Quantities.*

THE Queſtions in Multiplication of compound Quantities, are reduced to theſe three. 1. To multiply a Line *Plus* another Line, by a Line more another Line, as $a+b$ by $f+g$. 2. To multiply a Line *Minus* another Line, by a Line *Plus* another Line, as $a-b$ by $f+g$. 3. To multiply a Line *Minus* another Line, by a Line leſs another Line.

RULES for theſe Three CASES.

RULE 1. WHEN *the two Quantities given to be multiplied together, have each the Sign +, their Product ought to have the ſame Sign +.*

To multiply $a+b$ by $f+g$, the Operation muſt begin by multiplying a by f, writing af, which ſignifies the Product of a by f, making in like Manner as many particular Multiplications as there are Letters, it will be $af+bf+ag+bg$, which is the Product of $a+b$ by $f+g$. Let $a+b=AC$. Let $a=AB$, and $b=BC$, alſo let $f+g=AG$, and $f=AH$, and $g=HG$, I ſuppoſe that ACEG is a Rectangle, cut by Parallels which make the Parallelograms ABIH, FGIH, BCDI, DEFI, which ſeveral Parallelograms are together equal to their Sum ACEG, *the Whole being equal to its Parts*; now theſe four Products $af+bf+ag+bg$, are equal to theſe four Parallolegrams, as is evident, they are therefore equal to $AC\times AG$, *that is*, to the Parallelogram of AC by AG, or of $a+b$ by $f+g$.

RULE 2. MORE *into leſs, or leſs into more, gives a Product which ought to have the Sign —.*

THAT is, if one of the two Quantities hath the Sign —, and the other the Sign +, their Product ought to have the Sign —, to multiply according to this Rule, $a+b$ by $f-g$. 1. Multiply $a+b$ by f, which makes $af+bf$, but as $a+b$ ſhould not be multiplied by the whole Value of f by the Product $af+bf$ is too much the Value of g, that is, by $ag+bg$ I therefore take away what I had put too much, and the Expreſſion will be $af+bf-ag-bg$; let $a=AB$, or HI, $b=BC$, or ID, $g=HG$ or IF, and $f=AG$, hence $a+b=AC$, and $f-g=AH$; then $af=AB\times AG$, and $bf=BC\times BF$, it muſt be demonſtrated, that $af+bf-ag-bg=ACDH$, or that $ACDH=ACEG-FGHI-DEFI$, which is evident.

RULE 3. LESS *into leſs gives the Sign +.*

IN multiplying theſe two compound Quantities together, $a-b$ and $f-g$, their Product is $af-bf-ag+gb$, in which you ſee that the Product of $-g$ by $-b$, is marked with the Sign +, to prove that this is right. Let $a=AC$, and $b=BC$, hence $a-b=AB$, let alſo $f=AG$, and $g=HG$, hence $f-g=AH$, to demonſtrate what is required; conſider that as ABIH is equal to ACEG, or af. if there was ſubtracted, 1. DEGH, or ag, and BCEF, or bf, provided that DEFI, or bg, which was taken away too much be reſtored thereto, for in deducting DEGH and

BCEF, twice DEFI, or bg, is deducted, hence $af-bf-ag+bg$ is the Product of $a-b$ by $f-g$; therefore leſs into leſs gives +, *that is*, at the End of the Product there ought to be the Sign +, becauſe that having taken away too much, the overplus muſt be returned.

Division *of compound Quantities.*

9 The general Rule for Diviſion, whether the Quantities be ſimple or compound, is to put the Dividend above a Line and the Diviſor under the ſame, thus to divide $ax+cd$ by $xd+cb$, I write $\frac{ax+cd}{xd+cb}$, it was ſhewn in ſimple Diviſion, that the ſame Letters being found both above and below muſt be cancelled, the ſame muſt be alſo done here, when they are found in each Part of the Dividend and of the Diviſor. This Expreſſion $\frac{ax+dx}{bx+cx}$, may therefore be reduced to this $\frac{a+d}{b+c}$, which is ſimpler and of an equal Value.

As Diviſion undoes what Multiplication does, the three Rules that were given in Multiplication, ſuffice to ſhew what Signs the Quotient ought to have, 1. Whether the Dividend and Diviſor have both the Sign +, or 2. Whether one be +, and the other —, or 3. Whether both have the Sign —.

SECT. II.

Of the Power of Lines.

Definition 1.

10 *The firſt Power of a Line, is the Line itſelf before it is multiplied.*

11 Def. 2. The *ſecond Power, or Square of a Line, is the Product of the Line multiplied by itſelf.*

The ſecond Power of b is bb, or b^2; the Number 2 denotes the two Dimenſions of bb. Take Notice, that there is a great Difference between b^2 and $2b$; for $2b$, ſhews that b has been added to itſelf, whereas b^2, ſignifies that it is a Square, or that b is multiplied by itſelf; which is very different, for Example 3 and 3, or 3 added to itſelf makes 9. When a Line is marked at its Ends with two Capital Letters as A and B, its Product or ſecond Power, which is the Square ABCD, is marked with four Letters, or elſe by two, which are at the Ends of the Diagonal as A and C, or the ſame Line AB joined to itſelf with the Sign of Multiplication thus $AB \times AB$, or in ſhort, a Line is drawn above AB and the Sign of the ſecond Power joined thereto, in this Manner $\overline{AB}^2$, thus $\overline{MN}^2$ is a Square whoſe ſide is MN, or which is produced by MN multiplied by MN.

ADVERTISEMENT.

EUCLID *and the ancient Geometricians took the Square for the firſt Power, which is eſteemed by the modern Geometers to be the ſecond. A poſitive Quantity before it be multiplied by itſelf, or by another, hath one Value, or one Power. Quantities which have the Sign* +, *are called Poſitive, and ſuch as have the Sign* —, *are called Negative.*

DEF. 3. *A Line which being multiplied makes a Power, is called the Root of that Power.* 12

THE ſquare Root of *bb* or b^2, is a Line equal to *b*, that is, to the ſide of a Square equal to *bb*.

ADVERTISEMENT.

WHEN *the Root of a Power cannot be expreſſed by Number, which happens in certain Caſes, as ſhall be demonſtrated, this Sign* $\sqrt{}$ *is put before it, with the Sign of the Power thus* $\sqrt[2]{}$, *if it is a ſquare Root; if it be a Cube Root thus* $\sqrt[3]{}$.

DEF. 4. THE *Cube, or third Power of a Line, is the Product which it makes, when it multiplies its Square.* 13

THE Square of *b* is *bb*, if *bb* is multiplied by *b*, which makes *bbb*, the Solid or Cube is the third Power of the Line *b*, the Cube *bbb* may be expreſſed thus b^3, the Number 3 denotes its three Dimenſions; b^3 is different from 3*b*, for 3*b* is *b* taken or added to itſelf three Times and b^3 is multiplied once by itſelf, which produces its Square *bb*, or b^2, and ſecondly, that Square is multiplied by *b*, which makes *bbb*, or b^3, the Number 4 taken three Times, makes 12: But the Square of 4, (*which is* 16) multiplied by its Root, (*which is* 4) makes 64. When a Line as AB, is marked at its Ends by two Capital Letters, its Cube or third Power may be expreſſed thus; AB × AB × AB, which ſhews, 1. That AB is to be multiplied by AB, which gives the Square of AB. 2. That the Square of AB is to be multiplied by AB, the Root of that Square, this Expreſſion is abbreviated by putting a Line above AB with a little Figure, ſhewing the Number of its Dimenſions; thus $\overline{AB}^{1}$ is the Cube of AB.

DEF. 5. EVERY *Quantity which is compoſed of the Product of two other Quantities, is called a Plane.* 14

THUS *bd* is a Plane, or plane Quantity, one of the Letters whereof denotes the Breadth, and the other the Length, that is, its two Dimenſions; AB×BC is a plane Quantity, or ſimply a Plane.

DEF. 6. THREE *Quantities multiplied together produce a Solid.* 15

THUS *bcd* is a Solid, whoſe three Letters denote the three Dimenſions thereof, the Product of two of theſe Letters denote the Plane or Surface, and the third denotes the Height, or Thickneſs by which the Plane hath been multiplied.

ADVERTISEMENT.

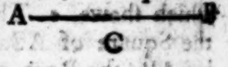
By what order ſoever two Quantities are multiplied, the Product is always equal: ab and ba make the ſame Plane; it is the ſame of three or more Lines: abc, bca, cab, by what order ſoever they are multiplied, the Product is the ſame. In *Euclid*, the Word Rectangle ſignifies a right angled Plane; the Planes and Solids hereafter treated of, are all ſuppoſed to be Rectangles.

17 Prop. 1. If *of two right Lines, one is cut into as many Parts you pleaſe; the Rectangles comprehended under the whole Line, and each Part of the divided Line, are equal to the Rectangle made by both the whole Lines.* Eucl. 2. Prop. 1.

Let AE and AB be two right Lines: AE is cut into the Parts AC, CD, DE, the Line AB is undivided; it muſt be proved, that the Rectangle made by the whole Lines AB and AE, is equal to the Rectangles made by the whole Line AB, and each of the Parts of AE; *that is*, that $AB \times AE = AB \times AC + AB \times CD + AB \times DE$. The Parts being together equal to their Whole, AE is the ſame as $AC + CD + DE$, conſequently 'tis the ſame thing, to multiply AB by AE, as by $AC + CD + DE$; therefore $AB \times AE = AB \times AC + AB \times CD + AB \times DE$; Q. E. D.

18 Prop. 2. If *a right Line be any how divided, the Rectangles comprehended under the whole Line, and each of the Parts, are equal to the Square of the whole Line.* Eucl. 2. Prop. 2.

Let the Line AB be cut into two Parts at the Point C, let $AB = a$, and $AC = b$, and $BC = d$, the Whole being equal to its Parts, $a = b + d$. therefore to multiply by a, and to multiply a by $b + d$, is doing the ſame thing; and conſequently $aa = ab + ad$; *that is*, the Square of the whole Line AB or aa, is equal to the Rectangles made by the whole AB, and its Parts AC and CB; *which was to be proved.*



19 Prop. 3. If *a Line be any how divided into two Parts, the Rectangle comprehended under the whole Line, and one of its Parts, is equal to the Rectangle of both the Parts, together with the Square of the firſt Part.* Eucl. 2. Prop. 3.

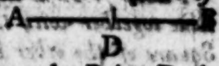
Let the Line AB be any how divided into two Parts at C, it muſt be demonſtrated, that $AB \times AC = AC \times BC + \overline{AC}^2$.† Let $AB = a$, and $AC = b$, and $BC = d$, hence as $AC + CB = AB$, ſo likewiſe $a = b + d$: then multiplying a and $b + d$ by the ſame Multiplicator b, the Products are equal; $ab = bd + bb$; now ab, is the Rectangle of AB by AC, and $bd + bb$ is the Rectangle made of the Parts b and d, together with the Square of the firſt Part b; Q. E. D.

Prop.

---

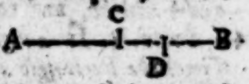
† *Sup. n.* 11.

20 PROP. 4. IF *a right Line be any how divided into two Parts, I ſay, that the Square of the whole Line, is equal to the Square of each Part, and to twice the Rectangle contained under both theſe Parts.* Eucl. 2. Prop. 4.



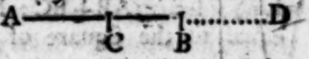
LET the Line AB be divided into two Parts at the Point D, it muſt be proved, that $\overline{AB}^2=\overline{AD}^2+2AD\times DB+\overline{DB}^2$. Let $AD=b$, and $DB=d$, then $AB=b+d$. Now the Square of $b+d$, is $b^2+2bd+d^2$,† but $\overline{AD}^2=bb$, and $\overline{DB}^2=dd$, and $2AD\times DB=2bd$, therefore $\overline{AB}^2=bb+2bd+dd$; *which was to be proved.*

21 PROP. 5. IF *a Line be divided into two equal Parts, and into two unequal ones, the Rectangle contained under the two unequal Parts, together with the Square of the intermediate Part, is equal to the Square of half the given Line.* Eucl. 2. Prop. 5.



THE Line AB is divided equally at C, and unequally at D, I ſay, that $AD\times DB+\overline{CD}^2=\overline{AC}^2$. Let AC or $CB=a$, and $CD=b$, then $BD=a-b$, and $AD=a+b$, now the Rectangle of $a+b$ by $a-b$, is $aa+ab-ab-bb$.‡ But $+ab-ab=0$; therefore the Rectangle is $aa-bb$. Conſequently $AD\times DB=aa-bb$, then adding to each Part bb or $\overline{CD}^2$ which is equal to bb, 'twill make $AD\times DB+\overline{CD}^2=aa$, *which was to be proved.* To add bb to $aa-bb$, is only to cancel $-bb$, for it is evident, that $aa-bb+bb=aa$, as has been explained.§

22 PROP. 6. IF *a right Line be divided into two equal Parts, and there be another right Line joined directly to it, if the firſt whole Line together with the Part added thereto conſidered as making only one Line therewith, be multiplied by the added Line, together with the Square of the whole firſt Line, the Rectangle or Product will be equal to the Square of half the firſt Line, together with the Line added thereto being conſidered as only one Line.* Eucl. 2. Prop. 6.

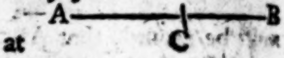


THE Line AB is divided into two equal Parts at the Point C, there is joined to it directly the right Line BD; it muſt be demonſtrated, that $AD\times BD+\overline{BC}^2=\overline{CD}^2$. Let AC or $CB=b$, then $AB=2b$, and $\overline{BC}^2$ or $\overline{AC}^2=bb$. Let $BD=d$, then $CD=b+d$, and $2b+d=AD$, therefore $AD\times BD=2bd+dd$; and $AD\times BD+BC=2bd+dd+bb$; now the Square of CD or $b+d$ is $bb+2bd+dd$;‖ hence $AD\times BD+\overline{BC}^2=\overline{CD}^2$; which was to be proved, *to wit*, that the Rectangle of AD × BD, together with the Square of AC or BC, is equal to the Square of CD.

---

† *Sup. n.* 8. ‡ *Sup. n.* 8. § *Sup. n.* 6. ‖ *Sup. n.* 8.

23 **Prop. 7.** If *a Line be any how divided, the Square of the whole Line together with the Square of one of its Parts, is equal to the Square of the other Part, and to twice the Rectangle contained under the whole Line, and the Part which was firſt taken.* Eucl. 2. Prop. 7.

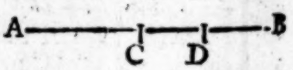


The Line AB being any how cut at the Point C. It muſt be demonſtrated, that $\overline{AB}^2+\overline{BC}^2=2AB\times BC+\overline{AC}^2$. Let $AC=b$, then $\overline{AC}^2=bb$; let $BC=d$, then $\overline{BC}^2=dd$; ſo likewiſe ſince $AB=b+d$, then * $\overline{AB}^2=bb+2bd+dd$, and $AB\times BC=bd+dd$, doubling theſe two Quantities $2AB\times BC=bd+2dd$, and adding to them $\overline{AC}^2$ or bb, it will become $2AB\times BC+\overline{AC}^2=2bd+dd+bb$, and conſequently equal to $2\overline{AB}^2+\overline{BC}^2$; Q. E. D.

24 **Prop. 8.** If *a right Line be any how divided into two Parts, four Times the Rectangle comprehended under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of the whole Line and the Part firſt taken conſidered as only one Line.* Eucl. 2. Prop. 8.

The Line AB was divided any how into two Parts at the Point C, it muſt be demonſtrated, that $4AB\times BC+\overline{AC}^2$, is equal to the Square of a Line equal to $AB+BC$. Let $AC=b$, and $BC=d$, then $AB=b+d$, and $AB+BC=b+d+d$, or $b+2d$, whoſe Square is $bb+4bd+4dd$,† which is equal to that of the Line $AB+BC$; now $AB\times BC=bd+dd$, ſince AB was ſuppoſed $=b+d$, and $CB=d$; then $4AB\times BC=4bd+4dd$, adding to each Part of $\overline{AC}^2$, its equal bb, it will be $\overline{AC}^2+4AB\times BC=bb+4bd+4dd$, which is the ſame Value as was found for the Square of the Line $b+2d=AB+BC$; hence the Square of $AC+4AB\times BC$ is equal to the Square of the Line $AB+CB$; which was to be proved.

25 **Prop. 9.** If *a right Line be divided into two equal Parts, and into two unequal ones, the Squares of the two unequal Parts are double the Squares of half the whole Line, and intermediate Part.* Eucl. 2. Prop. 9.



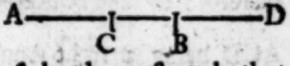
The Line AB is cut into two equal Parts at C, and into two unequal ones at D, it muſt be proved that $\overline{AD}^2+\overline{DB}^2$, is double $\overline{AC}^2+\overline{CD}^2$. Let AC, or $CB=b$, let $CD=d$, then $AD=b+d$, and $DB=b-d$. Hence $\overline{AD}^2=bb+2bd+dd$,‡ and $\overline{DB}^2=bb-2bd+dd$, and $\overline{AD}^2+\overline{DB}^2=bb+2bd+2dd-2bd$, and by reaſon that $+2bd-2bd$

---

* *Sup. n.* 8. † *Sup. n.* 8. ‡ *Sup. n.* 8.

make nothing, then $\overline{AD}^2+\overline{DB}^2=2bb+2dd$; which is the double of $\overline{AC}^2+\overline{CD}^2$; *which was to be proved.*

PROP. 10. IF *a right Line is divided into two equal Parts, and another right Line be directly joined to it, the Square of the whole Line with the Part adjoined conſidered as only one Line, together with the Square of the adjoined Part, is double the Square of half the whole Line, and Square of the ſaid half with the Part adjoined, eſteemed as only one Line.* Eucl. 2. Prop. 10. 26



THE Line AB is divided into two equal Parts at the Point C, and the Line BD is adjoined thereto, it muſt be demonſtrated, that the Square of AB+BD, together with that of BD, is double thoſe of BC and CD, that is, that $\overline{AD}^2+\overline{BD}^2$ is equal to $2\overline{BC}^2+2\overline{CD}^2$. Let AC or BC$=b$, then $\overline{BC}^2=bb$, and AB$=2b$; then $\overline{AB}^2=4bb$. Let BD$=d$; then AD$=2b+d$, and $\overline{AD}^2=4bb+4bd+dd$: Then the Square of AB+BD, together with the Square of BD, is equal to $4bb+4bd+2dd$; now the Square of BC, or bb, together with that of BC+BD, or $b+d$, is $2bb+2bd+dd$, which is the half of $4bb+4bd+2dd$; *which was to be proved*

PROP. 11. IN *every ambligonium Triangle, the Square of the Side which ſubtends the obtuſe Angle, is equal to the Squares of both the other Sides, together with a double Rectangle comprehended under the Side upon which the Perpendicular, (or Height of the Triangle) falls, and its Part produced ſo as to meet the Perpendicular.* Eucl. 2. Prop. 12. 27

THE Angle ACB of the ambligonium Triangle ABC is Obtuſe, and the Perpendicular AD falls upon the produced Side BC, which being ſo, it muſt be demonſtrated, that the Square of AB, the Baſe of the obtuſe Angle ACB, is equal to the Squares of both the other Sides AC and BC, together with twice the Rectangle made by the Side BC and its produced Part, comprehended between the obtuſe Angle C and the Perpendicular AD; which is expreſſed thus: $\overline{AB}^2=\overline{BC}^2+\overline{AC}^2+2BC\times CD$. ADB being a right Angle,† $\overline{AB}^2=\overline{BD}^2+\overline{AD}^2$, and by the ſame reaſon $\overline{AC}^2=\overline{CD}^2+\overline{AD}^2$: But $\overline{BD}^2=\overline{BC}^2+2BC\times CD+\overline{CD}^2$;‡ then ſubſtituting in the place of $\overline{BD}^2$; and in the place of $\overline{AD}^2+\overline{CD}^2$, the equal Quantity $\overline{AC}^2$, it will be $\overline{AB}^2=\overline{BC}^2+\overline{AC}^2+2BC\times CD$, *which was to be proved.* 28

PROP. 12. IN *every oxygonium or acute-angled Triangle, the Square of one of its Sides is equal to the Squares of both the other Sides, leſs twice the Rectangle comprehended under one of the ſaid other*

† *B.* 2. *n*, 142. ‡ *Sup. n.* 20.

other Sides, and one of its Parts contained between the Perpendicular which cuts it, and the Angle that's oppoſite to the Side firſt taken. Eucl. 2. Prop. 13.

I ſuppoſe that the Triangle ABD is an Oxygonium, that the Perpendicular AC falls upon the Side BD, it muſt be demonſtrated, that the Square of AB is equal to the Squares of both the other Sides AD, and BD, leſs twice the Rectangle made of the whole ſide BD, and its Part DC contained between the Perpendicular AC and the Angle D oppoſite to the Side AB firſt taken; it muſt therefore be demonſtrated that $\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2BD \times DC$. ACD being a right Angle, $\overline{AD}^2 = \overline{AC}^2 + \overline{DC}^2$,* taking away $\overline{DC}^2$ from each Side, it becomes $\overline{AD}^2 - \overline{DC}^2 = \overline{AC}^2$; this is done as has been taught, by cancelling $\overline{DC}^2$, where it is found with the Sign +, and writing it on the other Side with the Sign —, ſince ACB is a right Angle,† then $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$. Now $BC = BD - DC$, the Square of $BD - DC$ is $\overline{BD}^2 - 2BD \times DC + \overline{DC}^2$, hence $\overline{BC}^2 = \overline{BD}^2 - 2BD \times DC + \overline{DC}^2$; therefore putting in the Place of $\overline{AC}^2$ and $\overline{BC}^2$ the Quantities equal thereto, it will be $\overline{AB}^2 = \overline{AD}^2 - \overline{DC}^2 + \overline{BD}^2 - 2BD \times DC + \overline{DC}^2$: (and as $- \overline{DC}^2 + \overline{DC}^2 = o$.) $\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2BD \times DC$; Q. E. D.

S E C T. III.

Of the Ratio's and Proportions of Lines, Surfaces and Solids.

ADVERTISEMENT.

IT may be conſidered both what a Line is in itſelf, and alſo what it is in Proportion to others, which Proportion occaſions it to be eſteemed, equal, or unequal, ſhort, or long, it is the ſame of Surfaces, and Solids; now one Line may be proportionable to another, and they may be compared differently, (*viz.*) by conſidering either the Exceſs of one above the other, that is, their Difference, or how often one contains the other, which make two Sorts of Proportion. Geometers conſider only the latter, which they alſo call Ratio. which in general ſignifies Proportion, tis Ratio's, or this latter Sort of Proportion, that we are going to treat of, The Interpreters of Euclid formerly defined Ratio's, 28 *to be a Habitude of two Quantities of the ſame Species, compared one with another according to their Quantity.* When things are not of the ſame kind, they cannot be compared together; for Length cannot be compared with breadth, when the Ratio of two things is mentioned, it is then their Quantity or Bigneſs that is compared: But the word *Habitude* was here improperly uſed, the Greek Term uſed by *Euclid*, which is tranſlated *Habitude*, ſignifies what one thing is with reſpect to another; and that is what the word Ratio ſignifies in Geometry: But as I ſaid before, 'tis the ſecond Sort of Proportion that is here to be treated of; which will be explained by the following Definitions.

---

* *B.* 2. *n.* 142.

† *Book* 2. *n.* 142.

DEF. 1. THE *Ratio of a Line to a Line, of a Plane to a Plane, of a Solid to a Solid, is the Manner by which a Line contains or is contained in that with which it is compared, that a Plane contains, or is contained in a Plane, that a Solid contains or is contained in the Solid, with which it is compared.* 29

DIVISION ſhews how one Quantity is contained in another, hence to expreſs the Proportion, or Ratio of one Line to another, as of the Line A to the Line B. We divide B by A, writing them as has been ſhewn one under the other, $\frac{B}{A}$ or $\frac{A}{B}$: this Expreſſion ſhews or expreſſes the Ratio of A to B, *that is,* how many times, and in what manner A is in B, or what part B is of A, which is called the Exponent of the Ratio of A to B.

DEF. 2. *A Ratio whoſe Exponent is expreſſible by Number, is called a Ratio of Number to Number.* 30

THE Exponent ſhews either how one Quantity contains, or is contained in another, *that is, (in Arithmetical terms,)* it ſhews the Quotient of the two Quantities, when divided one by the other; if this Exponent or Quotient is a Number, as for Example. Suppoſe the Line B is contained ſix times in A, the Ratio of theſe two Lines A and B, is a Ratio of Number to Number.

DEF. 3. *A Ratio whoſe Exponent cannot be expreſſed by Number, is called Surd or Irrational.* 31

IF there cannot be found a Number, that ſhews exactly how many times the Line A contains, or is contained in the Line B, the Ratio of theſe two Lines A and B is ſurd. That there are ſuch Ratio's, ſhall be hereafter demonſtrated.

DEF. 4. THE *Equality of Ratio's is called Proportion.* 32

IF A has the ſame Ratio to B, as C has to D, the four Quantities are ſaid to be Proportional; which is expreſſed thus: A. B :: C. D. It has been ſaid, that the Ratio of A to B is expreſſed thus $\frac{A}{B}$; hence, that of C to D is in like-manner $\frac{C}{D}$. Conſequently the Proportion of theſe four Lines, which conſiſts in the Equality of their Ratio's, may be alſo expreſſed thus: $\frac{A}{B} = \frac{C}{D}$.

DEF. 5.

33 **Def. 5.** The *firſt Term of a Ratio is called* Antecedent, *and the other* Conſequent.

The Antecedent is the Thing which is related, or compared; the Conſequent is that with which the Compariſon is made. Every compariſon ſuppoſes two Terms: Ratio, which is a Relation or Compariſon, requires therefore two Terms.

34 **Def. 6.** *A Proportion hath two Antecedents, and two conſequents.*

Proportion is the Equality of the Ratio's that are compared. Every Ratio ſuppoſes two Terms: Therefore Proportion requires four.

35 **Def. 7.** The *ſame Term in a Proportion may be both Conſequent and Antecedent; and when it is ſo, the Proportion is called Continued.*

If A is to B as B to C, this makes two equal Ratio's, and conſequently makes this Proportion. A. B : : B. C.

Now in this Proportion B is conſequent to the firſt Ratio, and Antecedent to the Second. Which is expreſſed thus, ∺ A. B. C.

36 **Def. 8.** When *a continued Proportion has more than three Terms, 'tis called Progreſſion*

If A is to B as B to C, and B to C, as C to D, and C to D as D to E, and ſo on; it is called Progreſſion, which is expreſſed thus, ∺ A. B. C. D. E.

37 **Def. 9.** When *there are a certain Number of Quantities on each Side, for Example, three on one Side, and three on the other, if in comparing all the Six together thus, the Firſt with the Second, and the Fourth with the Fifth, the Second with the Third, and the Fifth, with the Sixth, they are in Proportion, it is called ordinate Proportion.*

Let A. B. C. be on one Side, and D. E. F. on the other; if A. B : : D. E and B. C : : E. F, they are in ordinate Proportion.

38 **Def. 10.** Three *Quantities being on one Side, and three on the Other: If in comparing the Firſt with the Second, and the Fifth with the Sixth, they are in Proportion, and that in changing the Order and comparing the Second to the Third, and the Fourth to the Fifth, they be alſo ſtill in Proportion, then it is called Perturbate, or Inordinate Proportion.*

Let there be $\left\{\begin{array}{l} \text{A. B. C.} \\ 12.\ 8.\ 4. \end{array}\right.$ and $\left\{\begin{array}{l} \text{D. E. F.} \\ 12.\ 6.\ 4; \end{array}\right.$

If A. B : : E. F and B. C : : D. E. this Proportion is called Perturbate or Inordinate, becauſe it does not obſerve the ſame Order.

Def. 11. The *firſt and laſt Terms of a Proportion, are called the Extremes of that Proportion; and the Second and Third, the Means.* 39

Let the Proportion be A. B :: C. D. the Extremes are A and D; and the Means are B and C.

Def. 12. The *homologous Terms of a Proportion, are thoſe which belong to the ſame Rank, or which have the ſame Name.* 40

In this Proportion A. B :: C. D. the Terms A and C which are the Antecedents, are Homologous, as are alſo B and D which are the Conſequents.

Evident Propoſitions concerning Ratio's and Proportions.

Prop. 1. Equal *Ratio's have equal Exponents.*

The Exponent of a Ratio ſhews how often one of the Terms is contained in the other, therefore if they are equally contained, their Exponents are equal. 41

Prop. 2. Equal *Quantities cannot be the Exponents of unequal Ratio's.* 42

Let X be the Exponent of the Ratio of A to B, and alſo of the Ratio of C to D, the two Ratio's are equal, ſeeing they equally contain each other..

Lemma. When *the firſt Term of a Ratio is multiplied by the Exponent of that Ratio, it becomes equal to the ſecond Term.* 43

Let A be the firſt Term or Antecedent of the Ratio of A to B, the Exponent of this Ratio, or *which is the ſame Thing*; the Quotient q of the Terms divided one by the other, ſhews how many times A is in B, or what part B is of A; conſequently being taken as many times as 'tis contained therein, *that is*, being multiplied by q, * it ought to become equal to B the Dividend, hence $Aq=B$; *which was to be proved.*

SCHOLIUM.

I ſhall henceforward generally ſuppoſe the Antecedent to be leſſer than the Conſequent; thus having mentioned A and the Conſequent B, I ſhall ſay that $Aq=B$, *nevertheleſs if it were otherwiſe, and that the Antecedent A, were greater than the Conſequent B, it would be equally true to ſay, that the Term A multiplied by the Exponent of A to B will always produce the Conſequent B, by reaſon the Exponent equally ſhews how one contains, or is contained in the other, whether the Ratio be ſurd, or of Number to Number. Hence when the Antecedent A is leſs than the Conſequent B, the exponent of that Ratio will be greater than Unit, and on the contrary leſſer, (which in the Arithmetical phraſe is called a Fraction)*

---

* *Sup. n.* 4.

on) when the Antecedent is greater than the Conſequent, but ſo that the Proportion be Ordinate, the firſt Term multiplied by the Exponent of its Ratio to the ſecond Term, will always produce the ſaid ſecond Term which was divided.

44 **Prop. 3.** Four *Terms will continue in Proportion howſoever changed, ſo long as the firſt Antecedent is to its Conſequent, as the ſecond Antecedent is to its Conſequent.* Which follows from the Definition of Proportion.

45 **Prop. 4.** Four *Quantities being proportional* Permutando, *that is, changing them and making the Antecedents become Conſequents, they ſtill continue Proportional.*

If $A.\ B :: C.\ D.$ It muſt be proved, that $B.\ A :: D.\ C.$ To contain and be contained are reciprocal Terms, thus if A contains B, as often as C contains D, and conſequently that there be an equality of Ratio's, it muſt be that B is contained in A, as often as D is contained in C, and that there is alſo an Equality of Ratio's, when a Conſequence is drawn from this Proportion, it is called a Concluſion *a Permutando*, or Inverſe Ratio.

46 **Prop. 5.** Four *Quantities being Proportional*, Alternando, *that is, in comparing the firſt and ſecond Antecedents together, and the firſt and ſecond Conſequents together, theſe four Quantities are alſo Proportional.* Eucl. 5. Prop. 16.

Let the Proportion be $A.\ B :: C.\ D.$ It muſt be proved, that *Alternatively*. $A.\ C :: B.\ D.$ let q be ſuppoſed to be the Exponent of the two Ratio's, * $Aq = B$, and $Cq = D$; then inſtead of $A.\ B :: C.\ D.$ it may be wrote $A.\ Aq :: C.\ Cq.$ It muſt then be proved, that *Alternately* $A.\ C :: Aq.\ Cq.$ which is evident, ſince that the Quotient of C, divided by A is $\frac{C}{A}$, the ſame as that of Cq divided by Aq, which is $\frac{Cq}{Aq}$. Now according to the Rules of Diviſion, † q may be Cancelled it being both above and below the Line, after which there remains only $\frac{C}{A}$; therefore the two Ratio's having one and the ſame Exponent, they are equal.

47 **Prop. 6.** If *to two Terms of a Ratio, be added two other Terms of the ſame Ratio, thus if to the Antecedent of the Firſt, be added the Conſequent of the Second, the ſame Ratio will continue.*

Let the Proportion be $A.\ B :: C.\ D.$ It muſt be proved that $A+C.\ B+D :: A.\ B.$ the Exponent of the Ratio's of A to B and of C to D is q. Hence $Aq = B$, ‡ and $Cq = D$. It muſt then be proved that $A+C.\ Aq+Cq :: A.\ B.$ The Quotient of $Aq+Cq$ divided by $A+C$, is q; § then the two Terms $A+C$ and $Aq+Cq$ having one and the ſame Exponent as A and Aq, or as A and B, they have the ſame Ratio. ‖

---

* *Sup. n.* 43. † *Sup. n.* 4. ‡ *Sup. n.* 43.
§ *Sup. n.* 4. ‖ *Sup. n.* 41.

PROP. 7. TWO *Terms of a Ratio being ſubtracted from two* 48 *other Terms of the ſame Ratio, thus Antecedent from Antecedent, and Conſequent from Conſequent, the ſame Ratio will remain.*

LET the Proportion be A. B :: C. D. It muſt be proved that A—C. B—D :: A. B. If *q*, is the Exponent of theſe Ratio's, then A*q*=B, and C*q*=D. [*] It muſt therefore be proved that A—C, A*q*—C*q* :: A. B. Now the Quotient of A*q*—C*q* divided by A—C, is *q*; then theſe two Terms have the ſame Exponent, and therefore the ſame Ratio as A and B.

PROP. 8. FOUR *Quantities being in Proportion,* Componendo, 49 *that is, the firſt Antecedent, more its Conſequent, is to its Conſequent, as the ſecond Antecedent more its Conſequent, is to its Conſequent.* Eucl. 5. Prop. 18.

A. B::C. D. It muſt be demonſtrated that A+B. B::C+D. D. 1. *Alternatively,* A. C :: B. D. [†] then adding the Terms C and D, (which are in the ſame Ratio,) [‡] to A and B. [§] A+B. C+D :: B. D. and again *Alternately* A+B. B :: C+D. D. Q. E. D.

PROP. 9. WHEN *ſeveral Quantities are in Proportion, the Sum* 50 *of the Antecedents, is to that of the Conſequents, as each Antecedent is to its Conſequent.* Eucl. 5. Prop. 12.

Suppoſe A. B :: C. D :: E. F. It muſt be proved that A+C +E. B+D+F :: A. B :: C. D :: E. F. By the preceeding Propoſition A+C. C :: B+D. D. *Alternately* A+C. B+D :: C. D. Now the Ratio of C to D is the ſame, as that of E to F, therefore A+C. B+D :: E. F. *Alternatively.* A+C. E :: B +D. F. then alſo according to the preceeding Propoſition A+C +E. B+D+F :: E. F or A. B, or C. D; for the Ratio is always the ſame, *which was to be proved.*

PROP. 10. FOUR *Quantities being in Proportion,* Dividendo, 51 *that is, the firſt Antecedent leſs its Conſequent, is to its Conſequent, as the ſecond Antecedent leſs its Conſequent, is to its Conſequent.* Eucl. 5. Prop. 17.

LET A. B :: C. D. It muſt be proved that A—B. B :: C—D. D. ſince A. B :: C. D. then *Alternately.* A. C :: B. D. then ſubtracting B from A, and D from C, [||] A—B. C—D :: A. C. Now the Ratio of A to C is the ſame as that of B to D; hence A—B. C—D :: B. D. then *Alternately* A—B. B :: C—D. D. *which was to be proved.*

PROP. 11. TWO *Quantities each having the ſame Ratio to a* 52 *Third, are equal one to another.* Eucl. 5. Prop. 9.

IF A. B :: C. D, it muſt be that A=C, for let *q* be the Exponent of the Ratio of A to B; it will be that of C to B, which is the ſame, then A*q*=B, and C*q*=B, [*] therefore A*q*=B=C*q*. theſe two Quantities being equal to a Third, *viz.* B, they are by Axiom the Third, equal to one another.

Prop. 12

---

[*] *Sup. n.* 43. [†] *Sup. n.* 46. [‡] *Sup. n.* 46.
[§] *Sup. n.* 47. [||] *Sup. n.* 8.
[*] *Sup. n.* 43.

PROP. 12. TWO *Ratio's equal to a third Ratio, are equal to*
53 *each other.* Eucl. 5. Prop. 11.

IF A. B :: E. F and C. D :: E. F, it muſt be proved that A. B :: C. D. If the Exponent of the Ratio of A to B is *q*, and that *q* be alſo that of the Ratio of E to F, which is the ſame, now the Ratio of C to D is the ſame, as that of E to F; therefore its Exponent is alſo *q*. The two Ratio's of A to B, and C to D: Have then one and the ſame Exponent; therefore they are Equal.*

PROP. 13. WHEN *two Quantities are multiplied by one and the*
54 *ſame Quantity, they continue in the ſame Ratio after the Multiplication, as they were in before.*

IF A and B are multiplied by *x*, it muſt be proved that A*x*. B*x* :: A. B. the Exponent of the Ratio of A to B is $\frac{A}{B}$, and that of the Ratio of A*x* to B*x* is $\frac{Ax}{Bx}$. Now the Exponent is juſt the ſame, for having cancelled *x* which is both above and below the Line in $\frac{Ax}{Bx}$, there remains $\frac{A}{B}$; † therefore the Ratio's having one and the ſame Exponent, they are equal. ‡

PROP. 14. TWO *Quantities being divided by a Third, the Quo-*
55 *tients thereof are in the ſame Ratio as the Quantities themſelves.*

LET the two Quantities be B and D, divide them by *x*, let the Quotient of B by *x* be called *p*, and let that of D by *x*, be called *q*. It muſt be proved that *p*. *q* :: B. D. Now $px=B$, and $qx=D$; then *px*. *qx* :: B. D. and *p* and *q* having been multiplied by *x*, according to the preceeding Propoſition *px*. *qx* :: *p*. *q*. then ſince *px*. *qx* :: *p*. *q*. it muſt be that *p*. *q* :: B. D. § Q. E. D.

PROP. 15. WHEN *four Quantities are Proportional, the Pro-*
56 *duct or Rectangle of the extremes, is equal to that of the Means.* Eucl. 6. Prop. 16.

A. B :: C. D. It muſt be proved that $A\times D=B\times C$, let *x* be the Exponent of the two equal Ratio's; then $Ax=B$, and $Cx=D$. Hence this Proportion may be thus expreſs'd: A. A*x* :: C. C*x*. It muſt then be demonſtrated that $ACx=ACx$, *which is evident.*

CORALLARY THREE *Quantities being in continued Proportion, the*
57 *Product of the Extremes is equal to the Square of the mean Term.* Eucl. 6. Prop. 17.

SUPPOSE ∺ A. B. C. Since A. B :: B. C. then $AC=BB$, or $AC=B^2$.

PROP. 16. WHEN *four Quantities are ſo diſpoſed, that the*
58 *Rectangle or Product of the Extremes is equal to that of the Means, they are Proportional.* Eucl. 6. Prop. 16.

---

* *Sup. n.* 41. † *Sup. n.* 4. ‡ *Sup. n.* 41.
§ *Sup.* . 53.

In theſe four Lines A, B, C, D the Product AD of the Extremes is equal to BC the Product of the Means. It muſt be demonſtrated that A. B : : C. D. let x be the Quotient of B divided by A, and z that of D divided by C: then $Ax=B$, and $Cz=D$. Hence I reduce the four Terms to theſe, A. Ax. C. Cz, According to the Suppoſition $A\times Cz=Ax\times C$, taking $A\times C$ from each Side, there will remain $z=x$; conſequently the Exponent of the Ratio of A to B, is equal to that of the Ratio of C to D; therefore they are Equal: * Conſequently the four Quantities are Proportional.

CORALLARY. 1. THE *four Terms of a Proportion being changed, will always be ranged Proportionally, provided that the two ſame Terms be always either the two Extremes or the two Means.* 59

LET the four Terms be A. B :: C. D. provided that A and D, for Example, be always either the two Means or the two Extremes, their Product is always equal, conſequently they are Proportional, according to this Propoſition.

CORALLARY 2. THE *Product of the mean Terms of a Proportion being divided by the firſt Term, the Quotient of the Diviſion will be the fourth Term; and if the ſame Product be divided by the fourth Term, the Quotient will be the firſt Term.* 60

FOR ,the Diviſor multiplied by the Quotient, makes a Product equal to the Dividend. †

DEF. THE *Product of two Quantities being equal to that of two others,(they may then make a Proportion,) if the Firſt is to the Third, as the Fourth is to the Second, the Proportion is called Reciprocal, or Inverſe.* 61

LET the two equal Rectangles be AC and EG, if AB. EF :: FG. BC, the Proportion is called *Reciprocal, or Inverſe,* which begins and ends in the ſame Figure. If one of the Sides of the two Rectangles is longer the other is Reciprocally ſhorter.

PROP. 17. IF *there are ever ſo many Quantities on one Side, and as many on the other, which being taken two and two are in the ſame Ratio, thoſe which are in equal Ratio, are Proportional.* Eucl. 5. Prop. 22. 62

LET there be three Quantities A, B, C on one Side, and three others D, E, F on the other Side. If A. B :: D. E and B. C :: E. F. it muſt be demonſtrated that A. C :: D. F. in equal Ratio. 1. *Alternately*, B. F :: A. D, and B. E :: C. F. ‡ then the two Ratio's of A to D, and of C to F being equal to that of B to E, they are equal: § Conſequently A. D :: C. F, and *Alternately* A. C :: D. F. *Q. E. D.*

PROP. 18. IF *the firſt Term is to the Second as the Third is to the Fourth, and the Fifth to the Second as the Sixth to the Fourth, the Sum of the Firſt and Fifth will be to the Second, as the Sum of the Third and Sixth is to the Fourth.* Eucl. 5. Prop. 24. 63

LET the ſix Quantities be A, B, C, D, E, F; if the firſt A is to the ſecond B as the third C is to the fourth D, and the fifth

E.

---

* *Sup. n.* 41. † *Sup. n.* 4. ‡ *Sup. n.* 46. § *Sup. n.* 53.

E to the ſecond B as the ſixth F is to the fourth D; which is expreſſed thus: A. B :: C. D and E. B :: F. D.

I ſay that A+E. B :: C+F. D; for, 1. *Alternately*, * A. C :: B. D and E. F :: B. D; then † A. C :: E. F; then A+E. C+F :: A. C. ‡ and ſince A. C :: B. D; then A+E. C+F :: B. D. § and *Alternately*, A+E. B :: C+F. D; *Q. E. D.*

PROPOSITIONS *concerning Proportions, which* EUCLID *in his fifth Book, hath expreſſed in a different Manner.*

EUCLID calls that Quantity a *Multiple*, which is meaſured exactly by a leſſer Quantity repeated a certain number of Times; and thoſe Quantities *Equimultiplies*, which contain the Quantities whoſe multiples they are, an equal number of Times, the Product of one Quantity multiplied by another, is the Multiple of that Quantity; thus Ax is the Multiple of A, but A and B having been multiplied by the ſame Multiplicator x, the two Products Ax and Bx are *Equimultiples* of A and B. The following Propoſitions have been demonſtrated above, the reaſon why they are again repeated, is their having been expreſſed in a manner different from that in Eucl. Book. 5.

64 PROP. 1. IF *there are ever ſo many Quantities, Equimultiples of as many other Quantities, each to each, as one is the Multiple of one, ſo are the whole the Multiples of the Whole.*

I expreſs this Propoſition thus: Bx and Cx being Equimultiples of B and C, I ſay, that the Quantity Bx is to B, and Cx to C, as Bx+Cx is to B+C, 1. Bx. Cx :: B. C. ‖ 2. *Alternately* Bx. B :: Cx. C. 3. Adding therefore the Quantities Cx and C, which are in the ſame Ratio, * to B and Bx it will be Bx. B :: Bx+Cx. B+C, and Cx. C :: Bx+Cx. B+C; *which was to be proved.*

65 PROP. 2. IF *the Firſt is the ſame Multiple of the Second, as the Third is of the Fourth, and the Fifth the ſame Multiple of the Second, as the Sixth is of the Fourth; the Sum of the Firſt and Fifth will be the ſame Multiple of the Second, as the Sum of the Third and Sixth, is of the Fourth.*

IF A. B :: C. D, and that E. B :: F. D. I ſay that A+E. B :: C+F. D. Since A. B :: C. D and E. B. F. D. then *Alternatively* A. C :: B. D, and E. F :: B. D; hence the Ratio of A to C is the ſame as that of E to F, theſe two Ratio's are the ſame as that of B to D; † therefore A. C :: E. F; adding E to A, and F to C, ‡ then A+E. C+F :: B to D.
66 Now *Alternately*, A+E. B :: C+F. D; *which was to be proved.*

PROP. 3. IF *the Firſt is the ſame Multiple of the Second as the Third of the Fourth, and there be taken Equimultiples of the Firſt and third; the Multiple of the Firſt will be the ſame Multiple of the Second, as the Multiple of the Third is of the Fourth.*

A

---

* *Sup. n.* 46. † *Sup. n.* 53. ‡ *Sup. n.* 47. § *Sup. n.* 53. ‖ *Sup. n.* 54.

* *Sup. n.* 47. † *Sup. n.* 53. *Sup. n.* 47.

A Quantity which is a Multiple contains the Quantity whoſe Multiple it is, therefore 'tis the ſame Thing as if it were ſaid, if the Firſt contains the Second, as many times as the Third contains the Fourth; therefore 'tis required to be demonſtrated that if A. B :: C. D, and that E and F are Equimultiples of A and C; *that is*, that E contains A as often as F contains C, and that therefore A. C :: E. F; it muſt be that B. D :: E. F. for ſince A. B :: C. D; then *Alternately*, A. C :: B. D. Now A. C :: E. F. Hence the Ratio of B to D being equal to one and the ſame Ratio, * it is the ſame as that of E to F. 67

Prop. 4. If *the Firſt is in the ſame Ratio to the Second as the Third is to the Fourth, the Equimultiples of the Firſt and Third, will alſo bear the ſame Ratio to the Equimultiples of the Second and Fourth, howſoever multiplied, if they are taken as they correſpond together.*

This is as much as to ſay, that if A. B :: C. D, likewiſe *Alternately*, A. C :: B. D. it muſt be that Ax. Cx :: Bz. Dz. For † Ax. Cx. :: A. C; and Bz. Dz :: B. D. But A. C :: B D. therefore ‡ Ax. Cx :: Bz. Dz. Q. E. D. 68

Prop. 5. If *one Quantity is the ſame Multiple of another Quantity, as the Part ſubtracted from one is of the Part ſubtracted from the other, the Remainder of one will be alſo the ſame Multiple of the Remainder of the other, as the whole was of the whole.*

This Propoſition may be expreſſed thus, if A. B :: E. F, I ſay that A—E. B—F :: A. B for then the Quantities which are ſubtracted from them are in the ſame Ratio, therefore they will remain in the ſame Ratio. § 69

Prop. 6. If *two Quantities are Equimultiples of two other Quantities, and there be Equimultiples ſubtracted from them, the Remainders are either equal thereto, or Equimultiples thereof.*

This Propoſition I expreſs thus. If Ax. Bx :: Ax—C. Bx—D. I ſay that Ax—C. Bx—D :: A. B. This is evident; for Ax. Bx. :: A. B. ‖ Now two Ratio's equal to a Third, are equal: * therefore if Ax. Bx. :: Ax—C. Bx—D; it muſt be that Ax—C. Bx—D :: A. B. 70

Prop. 7. Quantities *are one to another as their Equimultiples are one to another, if taken as they correſpond together.*

Bx. Cx :: B. C or Bx. B :: Cx. C; this has been proved. † The reſt of the Propoſition's of Euclid's fifth Book, will follow in their proper place.

---

* *Sup. n.* 53. † *Sup. n.* 54. ‡ *Sup. n.* 53.
§ *Sup. n.* 48. ‖ *Sup. n.* 54.
* *Sup. n.* 53. † *Sup. n.* 54.

SECT. IV.

Of compound Ratio's and their Properties.

Definition I.

71 Ratio's *whoſe Exponents are made by the Multiplication of two or more Exponents, are called Compound.*

Suppoſe x to be the Exponent of the Ratio of A to B, and that $x=zy$, then the Ratio of A to B is ſaid to be compounded of the two Ratio's, whoſe Exponents are z and y.

72 Def. 2. A *Ratio compounded of two equal Ratio's is called a duple Ratio.*

If the Exponent of the Ratio of A to B is made by z multiplied by z, *that is,* if zz be the Exponent of that Ratio. it is a duple Ratio.

73 Corallary Hence *a Ratio whoſe Exponent is a Square, is a duple Ratio.*

Let zz, or z^2, be the Exponent of the Ratio of A to B; it is compoſed of z multiplied by z, and therefore of two equal Exponents; it is therefore the Exponent of a duple Ratio.

74 Def. 3. *A Ratio compounded of three equal Ratio's is called a triple Ratio.*

75 Corallary Hence *a Ratio whoſe Exponent is a Cube, is a triple Ratio.*

If the Cube zzz, or z^3, is the Exponent of the Ratio of A to B, 'tis a triple Ratio, for its Exponent is made by the Multiplication of theſe three equal Quantities z, z, z.

76 Theorem 1 Several *Quantities being in Progreſſion, the Ratio of the Firſt to the Laſt, is compounded of the Ratio's of all the Quantities which are between the two Extremes.*

Let the Quantities be A,B, C, D, E,F, &c. It muſt be demonſtrated that the Ratio of A to C is compounded of thoſe of A to B, and of B to C, let x be the Exponent of A to B, then $Ax=B$; * that of B to C is z; then by the ſame Reaſon Bz, or $Axz=C$. Hence the three Quantities A,B,C. may be changed into theſe A, Ax, Axz. Divide Axz by A; the Quotient xz is the Exponent of the Ratio of A to Axz, which is made by the Exponents of thoſe Ratio's multiplied one by the other; wherefore by the firſt Definition, the Ratio of A to C is compounded of thoſe of A to B, and of B to C. By this method I can demonſtrate, that the Ratio of A to F, is compounded of all the Ratio's of the intervening Quantities.

77 Theorem 2. The *Ratio of two Planes is compoſed of the Ratio's which the Sides of One have to the Sides of the Other, of the Breadth to the Breadth, of the Length to the Length.*

---

* *Sup. n.* 43.

LET the two Planes be *ab* and *cd*, it muſt be demonſtrated that their Ratio is compounded of that of *a* to *c* and of *b* to *d*, let *z* be the Exponent of the Ratio of *a* to *c*; then $az=c$. Let *x* be that of *b* to *d*, then $bx=d$, and $azbx=cd$, Dividing *azbx* by *ab*, the Quotient will be *zx*. compoſed of *z* and *x* the Exponents of the Ratio's of *a* to *c*, and of *b* to *d*: Therefore the two Planes are in a Ratio compounded of thoſe of *a* to *c* and of *b* to *d*, *that is*, of their Sides Q. E. D.

ADVERTISEMENT.

IT muſt be remembered that it is here always ſuppoſed that the Planes and Solids are all Rectangles, therefore it was that this Remark was made before.

THEOREM 3. THE *Ratio of one Solid to another, is compounded* 78 *of the Ratio which the three Sides of one have to the three Sides of the other.*

LET the two Solids be *abc* and *def*; it muſt be demonſtrated that their Ratio is compounded of theſe three Ratio's $\frac{a}{d}$, $\frac{b}{e}$, $\frac{c}{f}$, the Ratio of *abc* to *def*, may be expreſſed thus $\frac{abc}{def}$. Now this Exponent is compoſed of the three Exponents of thoſe very Ratio's, of which it is required to prove that the Ratio of theſe two Solids is compounded, wherefore according to the firſt Definition, theſe two Solids are one to another in a compound Ratio to that of their Sides.

THEOREM 4. WHEN *four Quantities are proportional, the Product of the Antecedents is to that of the Conſequents, in duple Ratio* 79 *to that of each Antecedent to its Conſequent, or as the Square of each Antecedent is to the Square of its conſequent.*

LET *a*. *b* : : *c*. *d*; the Product *ac* of the Antecedents, according to the preceeding Propoſition, is to *bd* that of the Conſequents, in Ratio compounded of thoſe of *a* to *b* and of *c* to *d*, which being equal, the Ratio is duple. * The Square *aa* of the Antecedent *a* is to *bb*, the Square of the conſequent *b*, in Ratio compounded of the Ratio's of *a* to *b* and of *a* to *b*; and conſequently the double of theſe two Ratio's: Now theſe Ratio's are the ſame as the two Ratio's of *a* to *b* and *c* to *d*; conſequently the product *ac* is to the Product *bd*, as the Square *aa* is to the Square *bb*.

CORALLARY. SIMILAR *Planes, that is, thoſe whoſe Sides are proportional, are one to another in Ratio duple that of the Sides* 80 *of one to the Sides of the other.*

THEIR Ratio is compoſed of the Ratio's of the Sides of one to the Sides of the other; † theſe two Ratio's are equal, 'tis therefore

---

* *Sup. n.* 72.

† *Sup. n.* 77.

ſore a duple Ratio. * Hence all Squares being ſimilar Planes, they are in Ratio duple that of their Sides.

81 THEOREM 5. WHEN *ſix Quantities are proportional, the Product of the three Antecedents is to that of the three Conſequents, in triple Ratio of each Antecedent to its Conſequent, or as the Cube of each Antecedent to the Cube of its Conſequent.*

LET there be $a. b. c :: d. e. f.$ the Product abc is to the Product def, in Ratio compounded of thoſe of a to d, of b to e, and of c to f; † Now theſe three Ratio's are equal: This compound Ratio, is therefore triple. ‡ The Ratio aaa the Cube of a is to ddd the Cube of d in triple Ratio of that of a to d; now 'tis the ſame Ratio as thoſe of a to d, of b to e, and of c to f, conſequently the Products we are ſpeaking of are one to another, as the Cube of each Antecedent is to the Cube of its Conſequent.

82 CORALLARY. SIMILAR *Solids, that is, whoſe Sides are proportional, are in Ratio triple of the Sides of one to the Sides of the other.*

THE Sides of two ſimilar Solids are ſix proportional Quantities; therefore the Ratio of one to the other is triple. Hence all Cubes being ſimilar Solids, are in Ratio triple thoſe of their Sides.

83 THEOREM 6. IN *a Geometrical Progreſſion, the Ratio of two Terms between which are two other, interveneing ones, is duple, and if there are three intervening ones, it is Triple.*

LET ∺ A. B. C. D. the Ratio of A to C is compoſed or equal to a Ratio compoſed of that of A to B, together with that of B to C, § Now theſe Terms being in Progreſſion, the two Ratio's are equal; therefore the Ratio which they compoſe is a duple Ratio ‖ In the ſame Manner 'tis demonſtrated, that the Ratio of a to d is triple. *

84 THEOREM 7. WHEN *Quantities are proportional, their Squares are proportional.*

IF $a. b :: c. d.$ I ſay $aa. bb :: cc. dd.$ theſe Squares are in Ratio duple that of their Sides, † *that is*, in duple Ratio of a to b, and of c to d, and as theſe two Ratio's are equal by Suppoſition, thoſe which they compoſe are alſo equal, the Ratio of aa to bb, is then equal to that of cc to dd; therefore the Squares are proportional,

85 THEOREM 8. WHEN *Quantities are proportional, their Cubes are proportional.*

IF $a. b :: c. d.$ I ſay that $aaa. bbb :: ccc. ddd.$ for the Ratio of a^3 to b^3, and that of c^3 to d^3, are triple the ſame Ratio's, ‡ therefore they are equal.

86 THEOREM 9. THREE *Quantities being in continued Proportion, the Square of the firſt is to that of the ſecond, as the firſt is to the third.*

---

* *Sup. n.* 72. † *Sup. n.* 78. ‡ *Sup. n.* 74.
§ *Sup. n.* 76. ‖ *Sup. n.* 72.
* *Sup. n.* 74. † *Sup. n.* 80. ‡ *Sup. n.* 82.

LET there be ∺ $a . b . c$. I ſay that $a^2 . b^2 :: a . c$. the Ratio of a to c, is compounded of the two Ratio's of a to b and of b to c, ‡ theſe two Ratio's are equal; therefore the Ratio of a to c is duple, ‖ now the Ratio of a^2 to b^2 is alſo duple the ſame Ratio's; † then $a^2 . b^2 :: a . c$.

THEOR. 10. FOUR *Quantities being in continued Proportion, the Cube of the Firſt is to the Cube of the Second, as the Firſt is to the Fourth.* 87

LET them be ∺ $b . c . d . f$. The Ratio of b to f is compounded of the three intervening Ratio's, * and theſe three Ratio's being the ſame, the Ratio of b to f is triple, † now the Cube b^3 is to the Cube c^3 in a Ratio triple the ſame Ratio: ‡ Therefore $b^3 . c^3 :: b . f$.

THEOR. 11. IF *three Quantities and as many others taken two and two are in the ſame Ratio, and in perturbate Proportion, the Quantities that are in equal Ratio are Proportional.* Eucl. 5. Prop. 23. 88

THREE Quantities A, B, C, and others D, E, F taken two and two, are in the ſame Ratio, but the Proportion is perturbate, that is, that A. B :: E. F, and B. C :: D. E, it muſt be demonſtrated that A. C :: D. F. The Ratio of A to C is compounded of thoſe of A to B, and of B to C, alſo that of D to F is compounded of thoſe of D to E, and of E to F, now theſe two Ratio's are compounded of equal Ratio's; for A. B :: E. F, and B. C :: D. E. Conſequently the compounding Quantities being equal, the Quantities compoſed thereof are equal, hence A. C :: D. F. *Q. E. D.*

SECT. V.

OF the Compariſon of Ratio's.

PROBLEM.

TO *reduce two different Ratio's ſo as to have the ſame Conſequent.* 89

LET the two Ratio's be $\frac{a}{b}$ and $\frac{c}{d}$, to reduce them to the ſame Conſequent, I multiply the Antecedent and Conſequent of the Firſt by the Conſequent of the Second; which produces $\frac{ad}{bd}$; likewiſe multiply the Antecedent of the ſecond Ratio, and its

‡ *Sup. n.* 76. ‖ *Sup. n.* 72. † *Sup. n.* 80.
* *Sup. n.* 76. † *Sup. n.* 74.
‡ *Sup. n.* 82.

Conſequent by the Conſequent of the Firſt; which produces $\frac{bc}{bd}$; hence the two propoſed Ratio's are reduced to theſe $\frac{ad}{bd}$ and $\frac{bc}{bd}$. theſe two Ratio's have the ſame Conſequent, nevertheleſs there is the ſame Proportion between the Antecedent and its Conſequent, as there was before. §

90 THEOREM I. Two *Ratio's having one and the ſame Conſequent, are one to another as their Antecedents.*

LET the two Ratio's be thoſe of A to X, and of B to X, which have one and the ſame Conſequent X, it muſt be demonſtrated that they are one to another, as the Antecedents A and B, divide A and B by X, and the Firſt will be expreſſed thus $\frac{A}{X}$, and the Second $\frac{B}{X}$, now ‖A and B being divided by one and the ſame Diviſor X, the Quotients $\frac{A}{X}$ and $\frac{B}{X}$ are one to another as A and B; *Q. E. D.*

91 THEOR. 2. IF *two Quantities are unequal, the Greateſt has a greater Ratio to one and the ſame third Quantity, than the Leſſer, and the Ratio of that third Quantity to the Leſſer is greater than its Ratio to the Greateſt.* Eucl. 5. Prop. 8.

LET the two unequal Quantities be A and B, the Greateſt is A, and chuſe at pleaſure any third Quantity as C, it muſt be demonſtrated. 1. That the Ratio of A to C is greater than that of B to C, I expreſs the two Ratio's thus $\frac{A}{C}$ and $\frac{B}{C}$, or $\frac{AB}{C}$ they have one and the ſame Conſequent; therefore according to the preceeding Propoſition, they are one to another as A and B: Now A is greater than B; therefore $\frac{A}{C} > \frac{B}{C}$. 2. It muſt be demonſtrated that the Ratio of C to B is greater than that of C to A, that $\frac{C}{B} > \frac{C}{A}$. Reduce theſe two Ratio's to the ſame Conſequent, † thus $\frac{ACBC}{AB}$, the Ratio $\frac{AC}{AB}$ is then the ſame as $\frac{C}{B}$; and the Ratio $\frac{BC}{AB}$, the ſame as $\frac{C}{A}$; ‡ now theſe two Ratio's $\frac{AC}{AB}$ and $\frac{BC}{AB}$ are one to another, § as A is to B; but according to the Propoſition, A is greater than B; therefore the Ratio of C to B is greater, than that of C to A; *Q. E. D.*

Theorem

---

§ *Sup. n.* 54. ‖ *Sup. n.* 55. † *Sup. n.* 89.
‡ *Sup. n.* 54. § *Sup. n.* 90 and 54.

92 Theor. 3. Of *two unequal Quantities, that is the Greateſt, which hath the greater Ratio to a Third, and on the Contrary, that is the Leaſt, to which the ſaid third Quantity has the greater Ratio.* Eucl. 5. Prop. 10.

Let A and B be the two unequal Quantities, A having a greater Ratio to C, than that of B to C; thus $\frac{A}{C} > \frac{B}{C}$, and $\frac{C}{B} > \frac{C}{A}$. I ſay that $A > B$; *which muſt be demonſtrated.* 1. It is then ſuppoſed $\frac{A}{C} > \frac{B}{C}$, now * theſe two Ratio's are one to another as A and B; it muſt therefore be that $A > B$. 2. If $\frac{C}{B} > \frac{C}{A}$ It muſt be demonſtrated that B is leſſer than A, having reduced theſe two Ratio's to the ſame Conſequent $\frac{C}{B}$ to $\frac{CA}{BA}$, and $\frac{C}{A}$ to $\frac{CB}{BA}$, it muſt be according to what was ſuppoſed, that $\frac{CA}{BA} > \frac{CB}{BA}$, now theſe two Ratio's are one to another as A and B; † it muſt then be that A is the Greater and B the Leſſer.

93 Prop. 5. If *the Firſt is to the Second as the Third to the Fourth, and that the Third hath a greater Ratio to the Fourth, than the Fifth hath to the Sixth; the Firſt will alſo have a greater Ratio to the Second, than the Fifth hath to the Sixth.* Eucl. 5. Prop. 13.

Let the ſix Quantities be theſe A, B, C, D, E, F, the four Firſt of which are in Proportion A. B::C. D, or $\frac{A}{B} = \frac{C}{D}$: It muſt be demonſtrated that if $\frac{C}{D} > \frac{E}{F}$, the Ratio $\frac{A}{B}$ will alſo be greater, than the Ratio $\frac{E}{F}$; ſince $\frac{A}{B}$ and $\frac{C}{D}$ are one and the ſame Ratio, $\frac{C}{D}$ being greater than $\frac{E}{F}$, It muſt be that $\frac{A}{B}$ is alſo greater than $\frac{E}{F}$.

94 Prop. 6. If *the Firſt is to the Second as the Third is to the Fourth, and that the Firſt be greater than the Third; the Second will alſo greater than the Fourth; and if equal, equal; if leſſer, leſs.* Eucl. 5. Prop. 14.

Let the Proportion be A. B::C. D. It muſt be demonſtrated that if $A = C$, alſo $B = D$; and that if $A > C$, likewiſe $B > D$;

---

* *Sup. n.* 90. † *Sup. n.* 90.

and in ſhort if $A<C$, ſo likewiſe $B<D$. *Alternately*, A. C :: B. D; which cannot be, unleſs as $A=C$, that likewiſe B be alſo equal to D; or if A being greater than C, that B is likewiſe greater than D; or if A being leſſer than C, the Term B, be alſo leſſer than D.

95 PROP. 7. IF *three Quantities on one Side and three on the Other, being taken two and two, and that in an equal Ratio the Firſt is greater than the Third; the Fourth will be alſo greater than the Sixth; and if Equal, Equal; if Leſſer, Leſs.* Eucl. 5. Prop. 2.

LET A, B, C. be on one Side, and D, E, F. on the Other, let A. B :: D. E and B. C :: E. F. If $A>C$, I ſay that $D>F$. If $A<C$, that $D<F$. If $A=C$, that $D=F$. otherwiſe A is not to C, as D is to F, (*as was ſuppoſed*) if it be either greater or leſſer; for two unequal Quantities cannot contain a third or be contained therein, equally alike.

PROP. 8. IF *three Quantities on one Side, and three on the*
96 *Other, taken two and two, are in the ſame Ratio, they being in perturbate Proportion, and that in equal Ratio the Firſt is greater than the Third: The Fourth will be alſo greater than the Sixth; and if Equal, Equal; if Leſſer, Leſs.* Eucl. 5. Prop. 21.

LET there be A, B, C and D, E, F. If A. B :: E. F, (and B. C :: D. E, and that A is greater than C; I ſay that D is greater than F; if Equal, Equal; if Leſſer, Leſs: for C having been ſuppoſed leſſer than A; the Ratio of C to B is leſſer, than that of A to B: * then ſince B. C :: D. E, and by *Permutation*, C. B :: E. D, the Ratio of E to D is leſs than that of A to B, or than that of E to F, which is the ſame: D is therefore greater than F; † the reſt is eaſy.

PROP. 9. IF *four Quantities are in Proportion, the Sum of the*
97 *Greateſt and Leaſt, is greater than the Sum of the Other two.* Eucl. 5. Prop. 25.

LET the four proportional Quantities be A. B :: C. D. I ſuppoſe A the greateſt and D the leaſt, and I ſay that $A+C>B+C$. Let X be the exceſs of A above B, and let Z be the exceſs of C above D; hence $A=X+B$, and $C=Z+D$, in the Place of A and C I put what they are equal to, and I have this Proportion X+B. B :: X+D. D, which is the ſame as the Preceding one; it muſt then be demonſtrated that $X+B+D>B+Z+D$. Subtract from each part the Quantity B+D, hence it remains only to demonſtrate that $X>Z$. Seeing X is the difference between A and B, and Z that of C to D, $A-B=X$, and $C-D=Z$: Now A—B. B :: C—D. D. ‡ then putting X and Z in the Place of their equal Quantities, A—B and C—D, it will be X. B :: Z. D, or *Alternately* § X. Z :: B. D, and B. D ::

---

* *Sup. n.* 91. † *Sup. n.* 92. ‡ *Sup. n.* 51.
§ *Sup. n.* 46.

A. C, according to the Suppoſition, hence A being greater than C, alſo X⁊Z; *which remained to be proved.*

PROP. 10. QUANTITIES *compoſed of equal and unequal ones, are one to another as the unequal ones.* 98

LET the Quantities be theſe two *ab* and *ac*, they may be imagined as compounded of *b* and *c* multiplied by *a*; hence *ab*. *ac* :: *b*. *c*, ‖ ſo likewiſe *abd*. *acd* :: *b*. *c*.

CORALLARY HENCE *Triangles and Parallellograms having the ſame Altitude, are one to another as their Baſes.* Eucl. 6. Prop. 1: 99

THE Bigneſs of Triangles and Parallelograms * depends upon their Heights and Baſes; therefore if their Heights are the ſame, they are to each other as their Baſes.

‖ *Sup. n.* 54. * *B.* 2. *n.* 130 and 135.

* * *

BOOK III.

BOOK. IV.

SECT. I.

Of the Ratio's and Proportions of Lines.

According to the Idea which has been given of Ratio's and Proportions, it is evident, that in order to demonſtrate that four Lines are in Proportion, it muſt be ſhewn that if they are divided into two, three, or as many parts you pleaſe; if each part of the Firſt is Equal to each part of the Second, each part of the Third will alſo be Equal to each part of the Fourth; if the parts of the Firſt be greater than thoſe of the Second, thoſe of the Third will alſo be greater than thoſe of the Fourth; if Leſſer, Leſs; and that if each part of the Firſt be not contained exactly any certain number of times in the Second, each part of the Third will not be exactly contained in the Fourth, in Fine, that if there be Exceſs or Defect, in dividing the firſt and ſecond One by the Other, there will alſo be Exceſs or Defect in dividing the Third and the Fourth one by the Other. This is the moſt natural Method, ſeeing Ratio is only an expreſſion, which ſhews how one Quantity contains, or is contained in another; and it cannot be more plainly proved that a firſt Line is contained in a Second, as a Third is in a Fourth, than by proving it of four Lines.

1 Definition. A Parallel *Space is that which is between two Parallel Lines.*

X and Z being two parallel Lines, the Space which is between them is called a parallel Space.

2 Lemma 1. If *a parallel Space, or the Perpendicular which meaſures it, be divided by parallel Lines, the oblique Lines contained in that Space will be divided into as many parts, as the Perpendicular.*

The Perpendicular D C is the Meaſure of the parallel Space between X and Z, the Line A B is Oblique, and contained in that Space; divide D C into three parts by drawing the two Parallels E and F; I ſay that the two Parallels do alſo cut the oblique Line A B into three Parts. The two Parallels E and F divide the whole parallel Space, which is between X and Z, into three parallel Spaces, thro' which the oblique Line A B paſſeth, as well as DC, it is therefore divided into as many parts, as the Perpendicular DC; *which was to be proved.*

Lemma. 2. Oblique *Lines, which make the ſame Angles in equal Parallel Spaces, are equal, and equally Oblique.* 3

Let Z and X be two equal parallel Spaces, in which the oblique Lines B C and E F make the equal Angles BCH and EFG; I ſay that the two Lines are equal, and equally Oblique. From the Points B and E, draw Perpendicularly the Lines BH and EG, which are equal; * therefore the Triangles BCH, and EFG are Rectangled and entirely equal; † and therefore BC and EF are equal Lines; as alſo CH and FG; conſequently ‡ BC and EF are equally Oblique; *which was to be proved.*

Lemma. 3. Oblique *Lines which make the ſame Angles in unequal parallel Spaces, are unequal; longer, as the Space is greater, and ſhorter, as the Space is leſſer.* 4

The Oblique Lines BC and FG make the ſame Angles, in the Spaces X and Z; the Perpendicular AB is longer than EF; therefore the Space X is greater than the Space Z. *It muſt be demonſtrated,* that the Oblique Line F G, is ſhorter than the Oblique Line BC. Upon BA let there be taken the part B equal to EF, and thro' H let there be drawn a Parallel to the Baſe AC; the two Triangles ABC and EFG, being right Angled, and the Angle BCA being equal to FGE, *by the Suppoſition* they are Equiangular; § the Angle BKH is equal to BCA, and BHK to BAC; ‖ therefore the Triangle BKH is Equiangular with BAC, and likewiſe with FGE. FE is ſuppoſed equal to BH; therefore FG =BK: * Now BK is part of BC; then FG its equal is alſo part of BC, and conſequently leſſer, Q. E. D.

Theorem. 1. Having *divided a parallel Space by two or more Parallels, the Perpendicular of that Space, and the oblique Line, are cut Proportionally.* 5

1. By *Lemma the firſt,* † the oblique Line is cut into as many parts as the Perpendicular, if for Example the Perpendicular is cut into ten parts, the Oblique Line will alſo be cut into ten parts. 2. *By the ſecond Lemma,* ¶ If the Parts of the Perpendicular are equal one to another, thoſe of the Oblique Line are alſo equal one to another, for the Oblique Lines make the ſame Angles with the Parallels; ‡ therefore they are equal and equally Oblique, according to this Lemma. If the ten Parts into which the Perpendicular is divided are all equal, the ten Parts of the Oblique Line are then alſo all equal. 3. *According to Lemma* 3. If the Parts of the Perpendicular are unequal, thoſe of the Oblique Line are alſo unequal. Hence it follows, that if there be ten equal

---

* *B.* 1. *n.* 65. † *B.* 2. *n.* 96. ‡ *B.* 1. *n.* 55.
§ *B.* 2. *n.* 80. ‖ *B.* 2. *n.* 27.
* *B.* 2. *n.* 96. † *Sup. n.* 2. ¶ *Sup. n.* 3.
‡ *B.* 2. *n.* 27.

Parts taken from the Perpendicular, and that there remain one Part which is either leſſer or greater, the Oblique Line will be ſo divided, that after ten equal Parts be taken therefrom, there will remain one which is leſſer, if the Remainder of the Perpendicular was leſſer, and greater, if the Remainder of the Perpendicular was greater, *as was proved in Lemma* 3. Whereſore as the whole is contain'd, or as the whole contains, ſo the Parts are contained, or do contain, therefore according to the Nature of Proportion, the two Lines we are treating of are cut Proportionally.

6 THEOR. 2. IF *ſeveral Oblique Lines are in one and the ſame parallel Space, and the Space be cut by a parallel Line, the Lines will be divided Proportionally.*

THE Oblique Lines EF and MN are between too Parallels, between which AC is Perpendicular. This Space is divided by a Parallel Z: then by the preceeding Theorem MN. AC::MD. AB. and likewiſe EF. AC::EG. AB, and *Alternately* MN. MD ::AC. AB::EF. EG. therefore MN. MD::EF. EG, and by *Permutation*, MN. EF::MD. EG; *which was to be proved.*

7 THEOR. 3 OBLIQUE *Lines which make the ſame Angles in different parallel Spaces, are one to another as thoſe Spaces.*

THE Oblique Lines BC and FG form the equal Angles BCA and FGE; conſequently if AB is equal to EF, by the ſecond Lemma * BC=FG. If AB is greater than EF, by the third Lemma † BC is greater than FG. If AB for Example is triple EF, then BC is triple FG; for ſuppoſing that BA is divided into three equal Parts; then by the firſt Lemma ‡ BC is alſo divided into three Parts, which by the ſecond Lemma, § are each equal to GF, for theſe parts make the ſame Angles; ‖ therefore they are equally Oblique: Therefore BC, is triple FG, Q. E. D. Thus it is demonſtrated that the Oblique Line FG is the ſame Part of the Oblique BC, as EF is of AB; or that FG is alſo contained in BC, as often as EF is contained in AB. If AB is equal to, or contains EF, one or more times, more ſome remainder, then BC is likewiſe equal to, or contains FG, exactly one or more times, more the ſame Remainder; therefore the Lines are equally Oblique, &c. *Q. E. D.*

---

* *Sup. n.* 3. † *Sup. n.* 4. ‡ *Sup. n.* 2.

§ *Sup. n.* 3. ‖ *B.* 2. *n.* 27.

SECT.

SECT. II.

Of the Ratio's and Proportions which the Sides of Triangles bear to each other.

Definition I.

8 Two *Triangles are Similar, when their Sides make equal Angles, or which are Equiangular.*

9 Def. 2. In *comparing ſimilar Triangles, the Sides which are oppoſite to equal Angles, are called* Homologous Sides.

X and Z are ſimilar or equiangular Triangles, each of their Sides as AB and DE which are oppoſite to the equal Angles C and F, are called Homologous, *that is,* proportional; we ſhall ſhew, that this is a proper Term thereto.

10 Theor. 2. The *Sides of two ſimilar Triangles are Proportional.* Eucl. 6. Prop. 4.

Thro' the Vertex of the two Triangles ABC and DEF draw Lines Parallel to their Baſes, and from their Vertices let fall upon the Baſes the Perpendiculars X and Z, the Angles ABC and DEF are equal, the Oblique Lines AC and DF form alſo the ſame Angles: Therefore AB. DE::X. Z::AC. DF. * Hence AB. DE:: AC. DF; by drawing Lines thro' D and E Parallel to the Sides AC and DF, it will be likewiſe demonſtrated that AB. DE:: BC. EF, and that two ſimilar Triangles have therefore all their Sides proportional.

'Tis *for this Reaſon that the correſponding Sides of ſimilar Figures are called* Homologous, *becauſe they are Proportional one to another, or have the ſame Ratio, which the Word* Homologous *implies.*

11 Theor. 2. Two *Triangles ſimilar to a Third are ſimilar one to another.* Eucl. 6. Prop. 21.

Let A, B, C, be three Triangles, if A and C are ſimilar to B, the Angles of B are equal to thoſe of A and C; then the Angles of A and of C are alſo equal one to the other, † therefore according to the firſt Definition, ‡ A and C are Similar to each other.

12 Theor. 3. If *the Sides of two Triangles are Proportional, they are Similar.* Eucl. 6. Prop. 5.

The Sides of the two Triangles ABC and DEF are ſuch, that AB. BC::FE. ED, and AC. AB::DF. EF; I ſay that theſe two Triangles are Similar. Upon EF deſcribe the Angle GEF equal to ABC, and the Angle EFG equal to BAC; therefore EGF is equal to ACB; § conſequently EFG and ABC are two ſimilar Triangles ; * it remains then only to ſhew, that the two Triangles EFG and EFD are equal ; and that as EFD, is the ſame as EFG, it is ſimilar to ABC. Seeing EFG and ABC are Similar, then AB. BC::EF. EG, by the Suppoſition, AB. BC::EF. ED ; then EG and ED, which have one and the ſame Ratio to EF, are equal. † The ſame way 'tis demonſtrated that all the Sides of EFG, are equal to thoſe of EFD ; which was to be proved.

---

* *Sup. n.* 7. † *B.* 3. *n.* 52, ‡ *Sup. n.* 8. § *B.* 2. *n.* 80.

13 THEOR. 4. Two *Triangles are ſimilar, when they have one Angle equal, and the Sides containing that Angle Proportional.* Eucl. 6. Prop. 6.

THE Angle FED is equal to ABC, and AB. BC::FE. ED ; I ſay that ABC and FED are entirely Similar, *to prove it.* Make the Triangle EFG Similar to ABC, the Triangles EFG and DEF are equal ; for AB. BC::EF. EG::EF. ED. as EG and ED have one and the ſame Ratio to EF ; they are therefore equal. ‡ Now ſince the Angle DEF is ſuppoſed equal to the Angle ABC, it is then alſo equal to FEG, which is made equal to ABC ; therefore the two Triangles DEF and GEF ; having two Sides equal, EG to ED, and EF to EF, and the Angles FED and GEF which theſe Sides contain equal, they are equal ; § then ſince DEF and CBA are Similar to EGF, they are Similar one to another. ‖

14 THEOREM 5. IF *two Triangles have one Angle of one equal to one Angle of the other, and the Sides of another Angle proportional, and the third Angle of each be of the ſame Sort,* that is, *Acute, Right, or Obtuſe, then are the two Triangles Equiangular ; and the Angles, whoſe Sides are Proportional, are equal.* Eucl. 6. Prop. 7.

LET the two Triangles be ABC and DEF, the Angle A is equal to the Angle D, and the Sides which contain one of the other Angles are Proportional ; I ſay, that if C and F are of the ſame Sort, the two Triangles are Equiangular ; and the Angles, whoſe Sides are Proportional, are equal. 1. Let C and F be Acute ; I ſay that the Triangles are Equiangular, (*viz.*) That the Angles B and E are equal, as alſo E and F. If B is equal to E, according to the preceeding Propoſition, the Triangles are Equiangular. But if B is greater than E, let ABG be made equal to DEF, * whoſe third Angle AGB is equal to the third Angle F, † and therefore Acute as that is, and ABG and DEF are Equiangular, and Similar ; therefore AB. BG::DE. EF. now it is ſuppoſed that DE. EF::AB. BC. accordingly AB. BG::DE. EF::AB. BC : then BC and BG having one and the ſame Ratio with AB, they are equal ; ‡ therefore the Triangle GBC is Iſoceles

---

* *Sup. n.* 8. † *B.* 3. *n.* 52. ‡ *B.* 2. *n.* 52.
§ *B.* 2. *n.* 98. ‖ *Sup. n.* 11.
* *B.* 2. *n.* 29. † *B.* 2. *n.* 76. ‡ *B.* 3. *n.* 58.

celes, and Conſequently the Angles BCG and BGC are Acute,* and Conſequently BGA is greater than a right Angle: † But it hath been demonſtrated that the Angle AGB is equal to the Angle F; which is ſuppoſed Acute, therefore it is at the ſame Time both greater and leſſer than a right Angle; *which is abſurd.* that if C and F are not Acute, then BGC equal to C is not Acute; which is falſe ‡. conſequently the Angles ABC and E muſt be equal, hence ABC and DEF are entirely Equiangular, § therefore if in the two Triangles, &c.

THEOREM 6. IF *two ſimilar Triangles have one Point common, and the Homologous Sides Parallel, the two other Sides meet directly, that is, they are in a direct Line with each other.* Eucl. 6. Prop. 32. 15

LET the two ſimilar Triangles be ABC, DCE, which have a Point common (*viz.*) C, the Sides AB and DC are Parallel, as are alſo AC and DE; I ſay that BC+CE is a right Line. Seeing AB and DC are Parallel, the Angle BAC=ACD, ‖ and the Angle ABC=DCE per Suppoſition; then the three Angles ACB, ACD, DCE, are equal to the three Angles ACB, CAB, ABC of the Triangle ABC, which together make two right Angles; * therefore BCE is a right Line. †

THEOREM 7. WHEN *two ſides of a Triangle are cut by a Line Parallel to the Side which ſubtends the Angle which they contain, they are cut Proportionally.* Eucl. 6. Prop. 2. 16

LET the Triangle be ABC, draw EF Parallel to BC. I ſay that AE. EB::AF. FC. the Triangle AEF is ſimilar to the Triangle ABC, ſeeing they are Equiangular; ‡ for beſides that the Angle A is common, the Angle AEF=ABC, and the Angle AFE=ACB: § then AB. AE::AC. AF, ‖ and *dividedly,* * AB—AE. AE::AC—AF. AF, now AB—AE=EB, and AC—AF=FC; then EB. AE::FC. AF, and *Permutatively,* AE. EB ::AF. FC.

THEOR. 8. IF *the Angle of a Triangle is cut equally in two by a right Line, which alſo cuts the Baſe, the Segments of the Baſe are one to another as the two other Sides, and if they be ſo, the Line drawn from the Vertex to the Baſe divides the Angle equally in two.* Eucl. 6. Prop. 3. 17

LET the Triangle be ABD, the Line BC drawn from the Vertex, cuts the Angle B equally in two, 'tis required to be proved, 1. That AB. BD::AC. CD. let there be drawn DE Parallel to BC, and produce the Side AB, until it meet the Parallel DE, the Angles ABC and AED are equal, † and by the ſame Reaſon CBD and BDE are equal. ‡ it is ſuppoſed that ABC and CBD are equal: therefore CBD and BED are alſo equal, and ſo conſequently

* *B.* 2. *n.* 84. † *B.* 2. *n.* 17. ‡ *B.* 2 *n.* 84.
§ *B.* 2. *n.* 80. ‖ *B.* 2. *n.* 25.

* *B.* 2. *n.* 75. † *B.* 2. *n.* 80. ‡ *Sup. n* 8.
§ *B.* 1. *n.* 27. ‖ *Sup. n.* 10.

* *B.* 3. *n.* 51. † *B.* 2. *n.* 27. ‡ *B.* 2. *n.* 23.

quently are BDE and BED. therefore the Triangle DBE, being Iſoceles, BD=BE. now AB. AC::BE (or BD.) CD. § then AB. AC::BD. CD. and *Alternately* AB. BD::AC. CD. 2. It muſt be proved that if all this be ſo, then BC divides ABD equally in two, ſince Suppoſing as above DE Parallel to BC, and AB produced to E, then AB. BE::AC. CD, * and by the Hypotheſis AB. BD::AC. CD. then BE=BD, † and the Triangle DBE is Iſoceles, ‡ and hath the Angle BED=BDE ; § but on account of the Parallels BC, ED, the Angles ABC, BED or its equal BDE are equal ‖ as are alſo CBD, BDE, * therefore the Angle ABC=CBD ; *which was to be proved.*

18 DEF. 1. THOSE *Lines which make the ſame Angles with the Lines which they croſs but which are taken from the other Side are called* Antiparallels.

PARALLEL Lines make with the Lines which they croſs, the ſame Angles taken on the ſame Side, □ thus if AFE=ABC the Lines FE and BC are Parallel. But if AFE=ACB the Lines are Antiparallels.

19 THEOR. 9. WHEN *the Sides of a Triangle are cut by Lines Antiparallel to its Baſe, they are cut in reciprocal Proportion.*

FE is Antiparallel to BC; therefore AFE=ACB, and AEF=ABC. the Triangle AEF hath then the ſame Angles as ABC, and conſequently they are Similar, and their Sides are Proportional, △ but their homologous Sides have not the ſame Poſition, for AB is not Homologous with AF, but with AE ; therefore the Sides AB and AC are not divided in direct Proportion. AB is not to AF as AC is to AE, ſo that of theſe four Quantities, the Firſt is to the Fourth as the Third is to the Second, AB. AE:: AC. AF.

20 PROBLEM 1. To *divide a right Line in the ſame Manner as one that is already divided.* Eucl. 6. Prop, 10.

THE Line AD is divided into the three Parts AB. BC. CD, 'tis required to divide the Line AG into three Parts proportional to thoſe of AD ; join AG to AD ; ſo as to make any Angle you will, then draw a right Line thro' the Points G and D, and draw Lines Parallel thereto thro' the Points B and C of the divided Line ; which Parallels divide AG in three parts, which are Proportional to thoſe of AD : for † AD. AC::AG. AF, and AC. AB::AF. AE ; therefore what was required is done.

21 PROBLEM 2. To *divide a right Line into as many equal Parts you pleaſe.*

THE given Line is X, required to be divided into three Parts. Join thereto an indeterminate Line, making therewith any Angle at pleaſure as ZAX, with any extent of the Compaſſes, and one

---

§ *Sup. n.* 16. * *Sup. n.* 16. † *B.* 3. *n.* 52.
‡ *B.* 2. *n.* 57. § *B.* 2. *n.* 83.
‖ *B.* 2. *n.* 27. * *B.* 2. *n.* 25. □ *B.* 2. *n.* 27.
△ *Sup. n.* 10. † *Sup. n.* 16.

Foot in A, mark upon Z with that extent three equal Parts, then from B the end of the three Parts, draw a Line to the Point C, the End of X, and thro' the Diviſions of Z, draw Parallels thereto, then according to what was demonſtrated in the preceeding Problem, AC will be divided like AB, that is, into three Parts.

PROB. 3. To *cut off from a given right Line any Part required.* Eucl. 6. Prop. 9. 22

IF from AC, *fig. preceed.* It was required to cut off the third Part, you need only divide AC into three Parts, according to the preeceding Problem, and take from AC one of theſe three Parts.

PROB. 4. Two *Lines being given, to find a third in Proportion thereto.* Eucl. 6. Prop. 11. 23

THE firſt Line is AB, the Second BC, join them together ſo as they may make only one right Line: then take AD equal to BC, which join to AB making any Angle therewith at Pleaſure; from D draw a Line upon B and thro' the Point C, draw a Line Parallel thereto, produce AD untill it meets the Parallel CE; which being done, I ſay that DE is the third Proportional ſought. For AB * is to AD, or to its equal BC, as AC is to AE; and therefore as † AC—AB is to AE—AD, that is, as BC is to DE; therefore ∺ AB. BC. DE; which was required.

PROB. 5. THREE *Lines being in Proportion, to find a fourth Proportional.* Eucl. 6. Prop. 12. 24

THE Firſt is AB, the Second AD, with which make any Angle at pleaſure, ſuch as BAD; the Third is BC which join to AB, ſo that they both together may make a right Line; then thro' B and D draw a right Line; and thro' the Point C draw the Line CE parallel thereto; prolong AD untill it meet the Parallel CE; which being done, I ſay that DE is the Fourth Proportional: for ‡ AB. AC::AD. AE; therefore *dividedly*, § AB. AC—AB:: AD. AE—AD; that is, that AB. BC::AD. DE, and conſequently *Alternately* AB. AD::BC. DE.

PROB. 6. To *find all the Lines that can be reciprocally Proportional to two given Lines.* 25

LET the two given Lines be AB and AC; the ſhorteſt AB, muſt be taken from the Longeſt AC, and a Circle be deſcribed having the longeſt for its Diameter; then from the Point B, the End of the ſhorter Line, draw upon the Diameter an indeterminate Perpendicular, as Y. Every Line drawn from A which cuts the Indeterminate Line, and ends at the Circle as AD, or which cuts the Circle and ends at the Indeterminate Line, will ſolve the Problem, AE and AD are reciprocal to AB, AC as are alſo AF and AG; for BE is Antiparallel to DC, ſince AEB=ACD, and

---

* *Sup. n.* 16. † *B.* 3. *n.* 48. ‡ *Sup. n.* 16.
§ *B.* 3. *n.* 48.

ABE=ADC: ‖ therefore the Lines are cut reciprocally. * AFC=ABG, † and ACF=AGE. ‡ then BG and FC are Antiparallels; therefore AC and AG are reciprocally divided.

26 **Prob.** 7. To *deſcribe upon a given Line a Figure Similar to a Figure given.* Eucl. 6. Prop. 18.

There is no more required than to reduce the given Figure into Triangles § and to draw Lines upon the given Line making the ſame Angles, * and forming Similar Triangles.

27 **Lemma** 1. If *from the right Angle of a Rectangled Triangle be drawn a Line Perpendicular upon the Baſe, it will divide it into two other Triangles Similar to the whole Triangle, and alſo one to another.* Eucl. 6. Prop. 8.

Let ABD be a Rectangled Triangle. If from the right Angle A be drawn AC Perpendicular upon the Baſe BD, I ſay that the three Triangles ABD, ABC, ACD are all Similar to each other, 1. They are all right Angled. 2. The Angle B is common to both the Triangles ABD and ABC; therefore all their Angles are equal, † and conſequently Similar. ‡ The Angle D, is alſo common to both the Triangles ABD and ADC; they are then alſo Similar, ſince ABC and ADC are Similar to a Third, they are Similar one to another; § conſequently the three rectangled Triangles ABD, BCA, CAD are Similar.

28 **Theor.** 10. If *a Line be drawn from the right Angle of a Rectangled Triangle, Perpendicularly upon the Hypothenuſe, it will be as follows.* (viz.) 1. *The Perpendicular is a mean Proportional, between the two Segments of the Hypothenuſe,* 2. *The longeſt Side of the Triangle is a mean Proportional between the Hypothenuſe, and the greateſt Segment.* 3. *The ſhorteſt Side of the Triangle is a mean Proportional between the Hypothenuſe, and the leſſer Segment.*

By the preceeding Lemma, the Rectangled Triangle ABD, is divided by the Perpendicular AC into two rectangled Triangles which are Similar thereto and alſo to one another; ſo that ABD, CBA, CAD are three ſimilar Triangles, therefore 1. ⸸ BC. AC∷AC. CD, or ∺ BC. AC. CD. 2. CD. AD∷AD. BD, or ∺ CD. AD. BD. 3. BC. AB∷AB. BD, or ∺ BC. AB. BD. *Q. E. D.*

29 **Prob.** 8. To *find a mean Proportional, between two given Lines.* Eucl. 6. Prop. 13.

Method 1. The two given Lines are AB and BC; join them ſo that they may make a right Line, from the middle thereof which is G, and with the Extent of AG or GC, deſcribe a Semi-circle, then upon B erect a Perpendicular, and produce it till it reaches the Circumference *to wit* at the Point D, I ſay that BD is a mean Proportional between AB, and BC.

‖ *B.* 2. *n.* 52. *and* 39. * *Sup. n.* 18. † *B.* 2. *n.* 52. *and* 39.
‡ *B.* 2. *n.* 39. *and* 53. § *B.* 2. *n.* 122.
* *B.* 2. *n.* 29. † *B.* 2. *n.* 80. ‡ *Sup. n.* 8.
§ *Sup. n.* 11. ⸸ *Sup. n.* 10.

The Angle ADC in the Semi circumference is a right Angle, * BD is Perpendicular by conſtruction; then by the preceeding Theorem BD is a mean Proportional between AB and BC; or ∺ AB. BD. BC.

Method 2. AB and CB are two given Lines; prolong AB ſo that BD=AC; and from D and A, as Centers, with the Extent of AB or CD; deſcribe two Circles which interſect in E, the Line EB or EC is the mean Proportional between AB and CB. By conſtruction AB=AE, and CE=BE: then EAB and CEB are both Iſoceles; and the Angle ABE is common to each of them, theſe two Iſoceles are then Equiangular, † and therefore Similar; ‡ then AB. BE∷BE. BC, or ∺ AB. BE. BC. §

Prob. 9. Having *the three firſt Lines of a Geometrical Progreſſion of Lines, to find others, ad inf.* 30

Let the three progreſſional Lines be, the Firſt AC, the Second AD, the third AB, divide AB into two equal Parts; and upon the middle thereof with the Extent of half AB, deſcribe a Semicircle, upon the Line AB take a Part equal to the firſt Line AC; and upon C erect a Perpendicular which is bounded by the Circumference at the Point D, from whence draw a Line to the Point B, and an indefinite Line X from A thro' D, alſo prolong the Line AB ad inf. which I call Z; then at the Point B erect a Perpendicular which cuts X at the Point E, from whence draw a Perpendicular which cuts Z at the Point F, from which erect a Perpendicular which cuts X at the Point G, from whence let fall a Perpendicular GH, and ſo on for the Reſt. 1. The Line AD being a mean Proportional between AC and AB, ‖ is equal to the ſecond Line given, which is hereby found, if it had not been given. 2. All the Triangles ADB, ABE, AEF, AFG, &c. are Equiangular, * ſince that beſide their common Angle XAZ, they are all right Angled; for the Angle ADB in the Semi-circle is a right Angle, † and EF and HG are Perpendicular upon X by conſtruction, as are DC. BE. GF upon Z; hence according to the Definition, ‡ they are Similar, and their Homologous Sides Proportional: § then AC. AD∷AD. AB, and AD. AB∷AB. AE, and AB. AE∷AE. AF, and AF. AG ∷AG. AH, &c. conſequently ∺AC. AD. AB. AE. AF. AG. AH, &c.

Plato's *Method of finding two mean Proportionals between two given Lines, mechanically.*

The given Lines are AB and AC, which join ſo as to make a right Angle. Place the Square X ſo that its Angular Point may be upon the produced Part of AB, and that one of its Sides be cloſe to C the End of AC. Z is another Square which muſt

O be

---

* B. 2. *n.* 44. † *B.* 2. *n.* 85. ‡ *Sup. n.* 8. § *Sup. n.* 10. ‖ *Sup. n.* 28.

* *B.* 2. *n.* 80. † *B.* 2. *n.* 44. ‡ *Sup. n.* 8. § *Sup. n.* 10.

be diſpoſed ſo that one of its Sides rub againſt X, and the other againſt the Point B the Extremity of AB: Hence the Triangles CDE and DEB, are right Angled, DA and EA are Perpendiculars, hence * ∺ AC. AD. AE, and ∺ AD. AE. AB; then ∺ AC. AD. AE. AB.

Des Cartes *his Method for finding between two given Lines as many Proportionals as you pleaſe.*

The Inſtrument he uſed is compoſed of ſeveral Squares, which are ſo adjuſted one to the other, that when the Angle FAE is ſhut, or that the two Rules FA and AE touch one another, all the other Rules BC. CD. DF. EF. touch each other and ſlide to the Point A; if the Angle EAF be opened, theſe ſame Rulers are impelled and driven back, therefore two Lines being given, puſh the Ruler BC, ſo that AB may be equal to the ſhorteſt Line, and open the Angle EAF ſo that the Ruler DE may be diſtant from A the Length of the ſecond Line. It is evident that AC and AD are two mean Proportionals between AB and AE. In order to find ſeveral mean Proportionals, the Number of Squares muſt be increaſed.

32 Theor. 11. The *Segments of two Chords which interſect within a Circle, are in reciprocal Proportion.*

Let the two Lines be BD, CE which interſect at the Point A. It muſt be proved that BA. CA::AE. AD. Let there be drawn BC, ED; which form the two ſimilar Triangles BAC, EAD; for the Angles at A are equal, and the Angle B is equal to the Angle E, ſeeing they are placed upon the ſame Arc CD, therefore their Homologous Sides are proportional. † Then BA. CA::AE. AD; then *Alternately*, AB is reciprocally to AE, as CA to AD; *which was to be proved.*

33 Lemma. 2. If *a Tangent and a Secant be drawn from a Point out of the Circle, the Tangent is a mean Proportional between the whole Secant, and the Part thereof which is out of the Circle.*

Let the Tangent be CB and the Secant CA; it muſt be proved that AC. CB::CB. CD, or ∺ AC. CB. CD. The Angle C is common to both the Triangles ACB and DCB, the Angle DBC is meaſured by half the Arc BD, ‡ the ſame half is alſo the Meaſure of the Angle BAC; § hence the two Triangles ACB and BDC having two Angles equal, and conſequently their Third, they are Similar, and Proportional; ‖ the Side DC of the Triangle BCD is Homologous to the Side BC of the Triangle ACB; therefore AC. BC::BC. CD, or ∺ AC. BC. CD. *Q. E. D.*

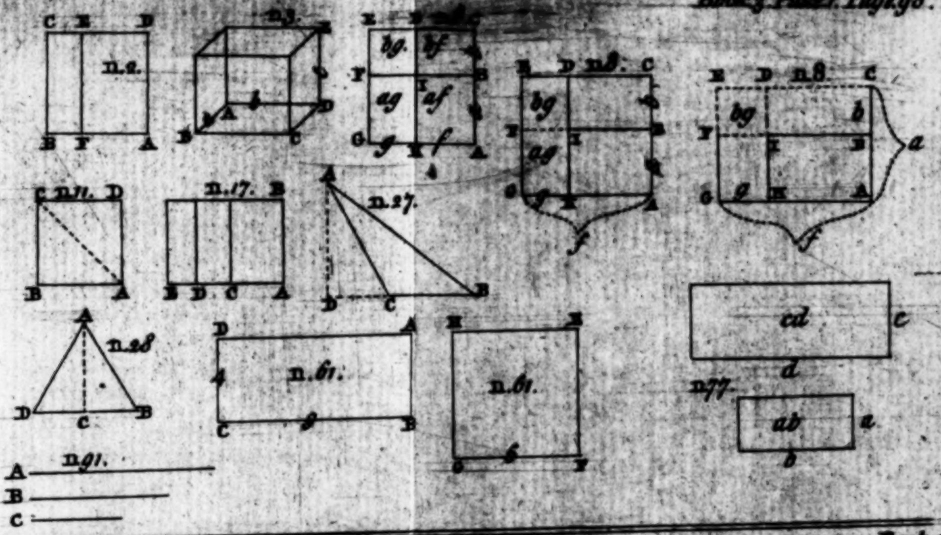
34 Prob. 10. To *divide a Line, ſo that its longeſt part may be a mean Proportional between its ſhorteſt part and the whole Line, which is called mean and extreme Ratio.* Eucl. 6. Prop. 30.

The given Line is BC, at the Point B erect a Perpendicular BG, equal to half BC, with the Extent of GB, deſcribe a Circle whoſe Diameter is conſequently equal to BC, draw the Secant

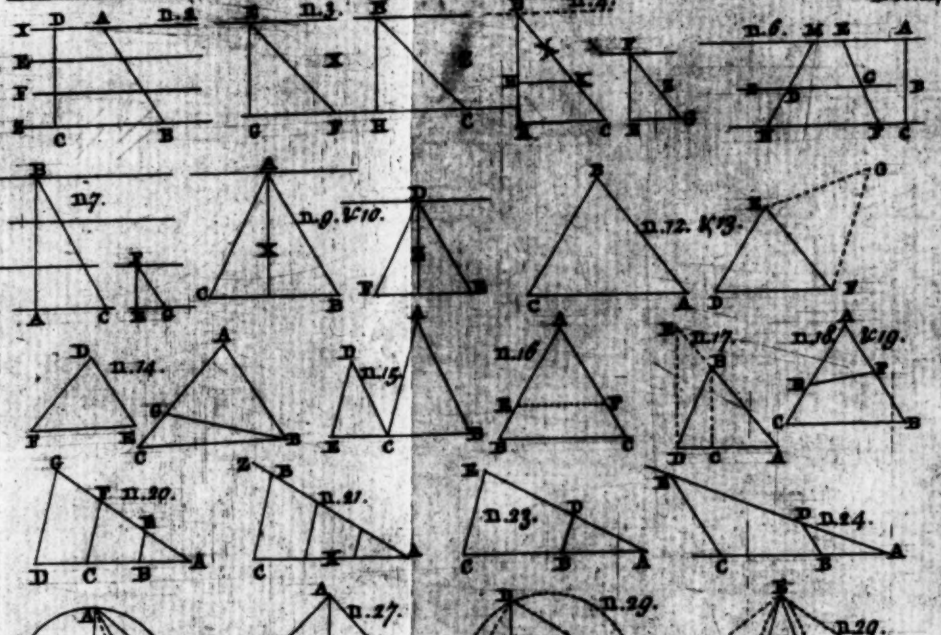
---

* *Sup. n.* 28. † *Sup. n.* 10. ‡ *B.* 2. *n.* 37. § *B.* 2. *n.* 39. ‖ *Sup. n.* 8 and 10.

Book 3. Plate 1. Page 98.



Book 4.



cant AC, then having taken CH upon BC, equal to CD, I ſay that the Line BC will be rightly divided at the Point H, as is required. If HC or DC is the Longeſt part, and BH or $BC-HC$ the Shorteſt, It muſt be demonſtrated that ∺ BC. CH. HB. by the preceeding Lemma, AC. BC::BC. CD, and ſubtracting the Lines BC and CD from AC and BC, the Remainders $AC-BC$ and $BC-CD$ are alſo in the ſame Proportion; * hence BC. CD::$AC-BC$. $BC-CD$. But by conſtruction $AC-BC=DC=HC$, and $BC-HC=HB$; therefore ∺ BC. CH. HB; *Q. E. D.*

Corallary 1. A *Line being divided into mean and extreme Ratio, if the mean or longeſt Part be added thereto, this new Line will alſo be divided into mean and extreme Ratio, whoſe mean is the ſaid firſt Line.* 35

Let AC be a Line divided into mean and extreme Ratio at the Point B, the mean or longeſt Part is BC, which add to the Line ſo as to make CD equal to BC, It muſt be proved that ∺ AD. AC. CD. Let $AC=a$, and $CB=x$; then $a-x=AB$, and $CD=x$, and $a+x=AD$. *It muſt then be proved* that ∺ $x.\ a+x.\ a.\ x.$ by the Hypotheſis $a.\ x::x.\ a-x.$ *Permutatively* $x.\ a::a-x.\ x.$ *Compoundedly*, $x+a.\ a::a-x+x.\ x.$ but $a-x+x$ $=$*Cypher*, therefore $x+a.\ a::a.\ x$, or ∺ $x+a.\ a.\ x$; *which was to be proved.*

Corallary 2. A *Line being divided into mean and extreme Ratio, if its ſhorteſt part be Subtracted from the Mean, the Remainder will be alſo divided into mean and extreme Ratio.* 36

The Line AC is divided into mean and extreme Ratio, at the Point B, its ſhorteſt part is AB, it muſt be proved, that if from BC be ſubtracted $DC=AB$, the Remainder AD will be divided according to the ſame Ratio; *that is*, that ∺ AD. AB. BD. Let $AB=y$, $BC=x$, $BD=x-y$. then $AC=y+x$, it muſt therefore be demonſtrated that ∺ $x-y.\ y.\ x.$ According to the Hypotheſis, $y.\ x::x.\ y+x.$ *by Permutation* $x.\ y::y+x.\ x.$ *dividedly*, $x-y.\ y::y+x-x.\ x$; as $+x-x=o$: Then $x-y.\ y::y.\ x.$ therefore ∺ $x-y.\ y.\ x$; *which was to be proved.*

Corallary 3. From the firſt Corallary it is eaſy to conclude that when a Line is divided into mean and extreme Ratio, innumerable other longer Lines may be divided into mean and extreme Ratio, by adding the mean to the whole firſt Line, and by the Second Corallary, an infinite number of ſhorter Lines may be divided into mean and extreme Ratio, by ſubtracting the ſhorteſt Part from the mean. 37

---

* *B.* 3. *n.* 48.

SECT. III.

Of the Ratio's and Proportions which the Circumferences of two or more Figures bear to each other, and to the Radii of their circumſcribing Circles.

Definition.

38 TWO *rectilinear Figures are called Similar, when their Angles are equal each to each, and the Sides containing them Proportional.*

39 CORALLARY. HENCE it follows that all regular Figures of the ſame Name are Similar; for being of the ſame Name they have the ſame Number of Sides and Angles; and becauſe they are regular, all the Angles are equal as well as the Sides containing them, * and therefore in an equal Ratio Proportional, therefore they are Similar, according to the Definition.

40 THEOR. I. THE *Circumferences of two ſimilar Figures are one to another in the ſame Ratio, as their Sides, each to each.*

LET there be two Hexagons X and Z, each Side of X is *a*; hence its whole Circumference is 6 *a*, each Side of Z is *b*, and conſequently its whole Circumference is 6 *b*; now 6 *a*. 6 *b*::*a*. *b*. †

41 THEOR. 2. THE *Circumferences of two regular and ſimilar Figures, are to each other as the Radii or Diameters of their circumſcribing Circles, and as their* Apotomes.

X and Z are two regular and ſimilar Polygons; from the Centers of their circumſcribing Circles, draw the Radii AB and AC, DE and DF; the two Triangles BAC and DEF are Iſoceles by conſtruction; and ſince that theſe two Figures are Similar, the Angles at their Centers BAC and EDF are the ſame; therefore the Triangles are Similar, then AB, or the Radius of X is to DE, the Radius of Z, as BC to EF. Now the Circumference of X is to that of Z by the preceedent Theorem, as BC is to EF, therefore the Circumference of X is to that of Z, as the Radius of X is to that of Z, or as the Diameter of one is to the Diameter of the other; for the Radii and Diameters have the ſame Ratio one to another, the Radii being each the half of the Diameters. If AM is Perpendicular upon BC, it is the Apotome of X, ‡ ſo likewiſe if DN is Perpendicular from D upon EF, it is the Apotome of Z; now according to what hath been proved, ed, * BC is to EF as the Apotome of X is to Z; therefore the Circumference of X is to that of Z, as the Apotome of X is to the Apotome of Z.

---

* *B.* 2. *n.* 106. † *B.* 3. *n.* 54. ‡ *B.* 2. *n.* 144.

CORALLARY, THE *Circumferences of two Circles are one to another, as the Diameters or Radii thereof.* 42

CIRCLES may be eſteemed as regular Polygons; now the Circumferences of two Polygons are one to another as their Diameters, therefore the Circumferences of Circles are one to another as their Diameters, and conſequently as the Radii thereof, ſeeing the Radii are the Halves of the Diameters.

THEOR. 3. THE *Arcs of an equal number of Degrees taken in different Circles, are one to another as the Circles whoſe parts they are.* 43

THE Degrees are proportional Parts of the Whole; they are therefore one to another as the Wholes whoſe parts they are, *that is,* as the Circles of which they are the Degrees.

THEOR. 4. THE *Chords of ſimilar Arcs, taken in different Circles, are one to another as the Arcs of which they are the Chords.* 44

RIGHT Lines being drawn from the Centers of the Circles to the Ends of the Chords, they will make ſimilar Triangles, ſince the Chords of ſimilar Arcs have their Angles at the Center equal; the Chords are then one to another as the Radii of the Circles; and therefore as the Circles, which are one to another as their Radii, † now by the preceeding Theorem, the Arcs of the ſame Number of Degrees are one to another, as the Circles of which they are the Parts, the Chords are therefore one to another, as the Arcs whoſe Chords they are.

THEOR. 5. IF *from the Point of Contact where ſeveral Circles touch, there be drawn a right Line croſſing the Circles, the Parts of that Line are one to another, as the Circles which it Interſects.* 45

THE Point where the ſeveral Circles touch is A, from A let there be drawn a Line croſſing the Circles C. D. E. It muſt be proved that the Parts of that Line AC. AD. AE. are one to another as the Circles which it interſects. Thro' A, draw the Tangent AB. hence the Angle BAE is meaſured by either the Arc CA, or AD, or AE, ‡ therefore theſe three Arcs are Similar; the Chords AC, AD, AE, of ſimilar Arcs, are by *the preceeding Theorem,* one to another as the Arcs AC. AD. AE, and theſe Arcs are one to another as their correſponding Circles; the Parts of the ſaid Line are therefore one to another, as the Circles which it croſſeth.

THEOR. 6. ANGLES *which are in the ſame or equal Circles, whether their Vertices be in the Circumference, or in the Center, are one to another as the Arcs upon which they are placed.* Eucl. 6. Prop. 33. 46

---

* *Sup. n.* 10. † *Sup. n.* 42. ‡ *B.* 2. *n.* 87.

This is very evident, by reaſon theſe Arcs are the Meaſures thereof. [*]

47 Lemma. One *of the acute Angles of a rectangled Triangle being divided into as many equal parts you pleaſe, and right Lines drawn upon the oppoſite Side, thoſe parts of that Side which are fartheſt from the Perpendicular are longeſt.*

Let ABE be a right angled Triangle, whoſe Angle BAE is divided into as many equal parts you pleaſe, by right Lines drawn from the Point A to BE. It muſt be proved that the parts of BE which are fartheſt from AB, are longeſt; that the Part DC is longer than CB, and ED longer than DC. Seeing, that the Angle BAC was by the Hypotheſis, made equal to the Angle CAD, the Angle BAD will be divided in two, and according to what has been demonſtrated, [†] DC. CB::AD. AB. Now AD >AB; [‡] then DC>CB. By the ſame Reaſon, if the Angle CAE is divided in two, as CAD is equal to DAE, then ED. DC::AE. AC; and ſince that AE>AC; then ED>DC; therefore the Parts of BE which are fartheſt from AB are longeſt.

48 Theor. 7. Of *two regular Polygons inſcribed in the ſame or different Circles, that which hath moſt Sides, has an Apotome longer or ſhorter in Proportion to its Circumference.*

Let the two Polygons be Z and Y, I ſuppoſe Y to have twice as many Sides as Z; that therefore if BC is a Chord of 20 Degrees, GH will be a Chord of 10, that is, if Z hath 18 Sides, Y hath 36. The Apotome of Z is AD, [§] and the Apotome of Y is FL; the Angle BAC is ſuppoſed double that of GFH; therefore DAC is equal to GFH, having divided the Angle DAC equally in two, the Angle DAE will be equal to LFH. The two Triangles LFH and DAE are right Angled, and have the Angles LFH, EAD equal; they are therefore Equiangular, [‖] and Similar; [*] hence LH. LF::DE. DA. [†] the Polygon Y has 36 Sides; hence 72 times LH is its circumference, multiply DE by the ſame Number. If DE were the fourth part of BC, then 72 times DE would be the exact Circumference of the Polygon Z, which hath 18 Sides, and contains 18 times BC; but *by the preceeding Lemma*, DE is ſhorter than EC, therefore leſs than the fourth Part of BC; then 72 DE is leſs than the Circumference of Z, which I call *m*, wherefore 72 DE hath leſſer Proportion to DA, but not *m*, which is longer, [‡] but as has been demonſtrated, LH. LF::DE. DA, alſo [§] 72 LH.LF:: 72 DE. DA; therefore 72 LH hath a leſs Proportion to the Apotome LF, than the ſaid *m*, the Circumference of the Polygon Z hath to its Apotome DA; *which was to be proved.*

The preceeding Demonſtration well underſtood, may be generally applied to all other Polygons, not only in multiple Ratio of their Sides, but alſo in any Proportion whatſoever, as 5 to 13, or what other you will, which cannot be only of Number to Number, hence inſtead of (as in the propoſed Example) the Angle DAC being double DAE, it will be here as 13 to 5, and it may be ſuppoſed as divided into 13 equal parts, of which the Angle DAE contains 5; and by the ſame Lemma, the five Portions of the Part DE which are neareſt to the Perpendicular DA, are always in leſſer Proportion, than the 13 of the Line DC; and therefore 26 times DE are always leſs, than ten times DC, which make the Circumferences of the Polygons, whoſe number of Sides is 13 and 5, DE and DC being each only half their Sides, and therefore the Concluſion will always be the ſame as before.

---

[*] *B.* 2. 10 and 39. [†] *Sup. n.* 17. [‡] *B.* 3. *n.* 59. [§] *B.* 2. *n.* 144. [‖] *B.* 2. *n.* 80,

[*] *Sup. n.* 8. [†] *Sup. n.* 10. [‡] *B.* 3. *n.* 91. [§] *B.* 3. *n.* 54.

CORALLARY 1. OF *two Polygons having the ſame Circumference, that which has the moſt Sides hath the longeſt Apotome.* 49

CORALLARY 2. THE *Radius of a Circle having the ſame Circumference as a Polygon, is longer than the Apotome of that Polygon.* 50

A Circle is a Polygon having infidite Sides, whoſe Apotome is its Radius, which is therefore longer than the Apotome of any other Polygon, according to the preceeding Propoſition.

SECT. IV.

OF the Ratio's and Proportions of Surfaces.

THEOREM 1.

IN *every Parallelogram, the adjacent Parallelograms, or thoſe which are about the Diameter, are ſimilar to each other, and alſo to the whole Parallelogram.* Eucl. 6. Prop. 24. 51

LET ABCD be a Parallelogram, whoſe Diameter is AC, about which there be two other Parallelograms AGFK and CEFK. 1. Each of them hath one Angle common with ABCD. The oppoſite Angles of the Parallelograms are equal; * therefore GAK=ECH=HFE, and KAG=KFG: Hence the three Parallelograms having two of their Angles equal, the others are alſo equal; for their Sides being Parallel, they make the ſame Angles. † The Triangles ABC, AGF, FEC, are therefore Equiangular, conſequently Similar, ‡ as alſo ADC, AKF, FHC; then § AG. GF::AB. BC, and AG, AF::AB, AC, and AF.

---

* *B.* 2. *n.* 119. † *B.* 2. *n.* 27. ‡ *Sup. n.* 8.
§ *Sup. n.* 10.

AK::AC. AD. ſo likewiſe AK. AG::AD. AB; therefore AKFG and ABCD are Similar; ſo likewiſe CEFH, and ABCD are Similar. *

52 THEOR. 2. IF *from a Parallelogram be taken away a Similar Parallelogram, and alike poſited in regard to the whole, and having one Angle common thereto, the Parallelogram taken away is about the Diameter of the whole Parallelogram.* Eucl. 6. Prop. 26.

LET the Parallelogram AGFE be ſubtracted from the Parallelogram ABCD, they are Similar, and have the Angle EAG common. It muſt be proved that their Diameters are in the ſame Line. If it be denied, and it be aſſerted that the Diameter AHC interſects EF at the Point H; draw HI, parallel to AE, the Parallelograms EI and DB are Similar, † therefore AE. EH::AD. DC::AE. EF. ‡ then EH=EF. § Hence the Part is equal to its whole; *which is abſurd.*

53 THEOR. 3. WHEN *a Line is divided into mean and extreme Ratio, the Rectangle of the whole Line and its leſſer part is equal to the Square of the mean Part.*

LET AB be a Line divided into mean and extreme Ratio, the mean Part is AD. then AB. AD::AD. DB, or ∺ AB. AD. BD. therefore $\overline{AD}^2$=AB×DB; ‖ *which was to be proved.*

54 CORALLARY. To *divide a right Line, ſo that the Rectangle of the whole Line and one of its parts, may be equal to the Square of the other part.* Eucl. 2. Prop. 11.

THIS is only to cut the given Line into mean and extreme Ratio, according to the Theorem.

55 THEOR. 4. IF *two right Lines interſect within a Circle, the Rectangle comprehended under the two Parts of the one, is equal to the Rectangle contained under the two Parts of the other.* Eucl. 3. Prop. 35.

THE two Chords BD and CE of the Circle X interſect at the Point A, it muſt be proved that BA×DA=CA×EA. It has been proved * that BA. AE::AC. AD. then BA×AD, the product of the Extremes, =AE×AC, the Product of the of Means. † *Q. E. D.*

56 THEOR. 5. IF *from a Point any where taken out of a Circle, be drawn two right Lines, one touching the Circle and the other cutting it, and bounded by the concave Circumference, the Rectangle comprehended under the whole cutting Line and its part that is out of the Circle, is equal to the Square of the Tangent, or touching Line.* Eucl. 3. Prop, 36.

FROM the Point C which is out of the Circle, let there be drawn two right Lines, CB which touches the Circle, and

---

* *Sup. n.* 38. † *Sup. n.* 51. ‡ *Sup. n.* 38. § *B.* 3. *n.* 52. ‖ *B.* 3. *n.* 57.

* *Sup. n.* 32. † *B.* 1. *n.* 56.

CA which cuts it. It muſt be proved that $AC \times CE = \overline{BC}^2$, it has been demonſtrated, * that ∺ AC. BC. DC: therefore $AC \times CD = \overline{BC}^2$; † *which was to be proved.*

CORALLARY 1. IF *from a Point taken at pleaſure out of a Circle, be drawn ever ſo many right Lines, which croſs the Circle and end at the concave Circumference; the Rectangle comprehended under any one of the cutting Lines, and its part out of the Circle, is equal to the Rectangle contained under any other cutting Line, and its part out of the Circle.* 57

FOR each of the Rectangles is equal to the Square of the cutting Line, which is drawn from the ſame Point.

CORALLARY 2. IF *from a Point taken at pleaſure out of a Circle, be drawn two Tangents, or two right Lines which touch the Circle, they are equal to each other.* 58

FOR the Square of each of the Tangents is equal to the Rectangle of a Secant Line and its part out of the Circle; and therefore theſe Squares are equal to each other; from hence it follows, that the Lines which are the Sides thereof, are equal.

THEOR. 6. IF *from a Point taken at pleaſure out of a Circle, be drawn two right Lines, one of which cuts the Circle and ends at the concave Circumference, and the other only reaches to the Circle; and that the Rectangle comprehended under the whole Secant or cutting Line and its part out of the Circle, be equal to the Square of that Line which only reached the Circle, then this laſt Line will be a Tangent to the ſaid Circle.* Eucl. 3. Prop. 37. *ſame fig.* 59

SEEING the Square of the Line which juſt reached the Circle, is equal to the Rectangle AC×DC, it is equal to the Tangent BC, whoſe Square is equal to that Rectangle, it cannot then be a different Line, therefore it is a Tangent: *which was to be proved.*

THEOREM 7. IF *from a Point in the Circumference of a Circle two Lines be drawn to its concave Circumference, and there be a third Line drawn croſs thoſe two Lines, to two Points in the Circumference, equally diſtant from the given Point, the Rectangle comprehended under one of the Lines and its part contained between the given Point and third Line, is equal to the Rectangle of the other Line, and its part contained between the ſaid given Point and third Line.* 60

LET K be the Point in the Circumference, from which let there be drawn the Lines KF and KG, and thro' the Points E and D which are equally diſtant from K, draw the Line ED, it muſt be proved that KF×KB=KG×KC. The two Lines BC and FG are Antiparallels; for the Angle KBC is meaſured by half the Arc EF, more half the Arc KD, or its equal KE; ‡ now the half of KF is alſo the Meaſure of KFG: § therefore

---

* *Sup. n.* 33. † *B.* 3. *n.* 57. ‡ *B.* 2. *n.* 52.
§ *B.* 2. *n.* 39.

$KBC=KGF$. by the ſame reaſoning $KCB=KFG$. Hence FG and BC are Antiparallels, according to the Definition thereof; * then KF. KC::KG. KB; † therefore $KF\times KB=KG\times KC$. ‡ *which was to be proved.*

61 CORALLARY. THE ſame Things being ſuppoſed as above, it not only follows that if from the Point K there be drawn Lines *ad. inf.* as KF or KG, which terminate at the concave Circumference, and are croſſed by the right Line ED, all the Rectangles made in the ſame Manner as above, are equal one to another, ſeeing theſe Lines taken two and two, are each equal to the Rectangle $FK\times KB$; but alſo, that if from the Point K be drawn the right Line KE, the Square of the ſaid Line KE will be equal to each of the aforementioned Rectangles; for having drawn the right Line EF, it will always form two ſimilar Triangles ſuch as FKE, and EKB, whoſe Angle K is common, and the two other Angles alſo equal; *viz.* BFK to KEB, and FEK to EBK, § each ſtanding upon equal Parts of the Circle's Circumference, hence their homologous Sides are Proportional; and ‖ therefore FK. KE::KE. * KB; then the Rectangle $FK\times KB$ is equal to the Square of KE, † and it alſo follows, that if the Line ED is a Diameter, the Line KE will be the Side of a Square inſcribed in the Circle, ‡ to which each of the Rectangles contained under the ſaid Lines, ſuch as $FK\times KB$ or others, are equal.

62 THEOR. 8. IF *a Quadrilateral be inſcribed in a Circle, the Rectangle contained under the Diagonals is equal to the Sum of the Rectangles made of the oppoſite Sides.*

LET the Quadrilateral be ABCD, whoſe Diagonals are AC and BD; it muſt be proved that $AC\times BD=BC\times AD+AB\times DC$. Draw BE, ſo that the Angle ABE may be equal to CBD, and that each of them being ſubtracted from the Angle ABC, the Remainders CBE and ABD may be equal; hence as the Angles ADB and ACB ſtand upon the ſame Arc, they are equal; § the Triangles BDA and BCE are Equiangular, ‖ and Similar; * therefore BD. AD::BC. CE; † hence the Rectangle of the Extremes is equal to that of the Means; and Conſequently $BD\times CE=AD\times BC$. ‡ By the ſame Reaſon, the Triangles BDC and BAE are Similar; for $ABE=DBC$ by conſtruction, and $BAC=BDC$ as ſtanding upon the ſame Arc BC, their Homologous Sides are then alſo Proportional; therefore BD. CD::AB. AE; hence $BD\times AE=CD\times AB$; § Now $BD\times AE\times BD\times CE=BD\times AC$, ‖ then ſeeing $BD\times AE=CD\times AB$, and $BD \times CE = AD \times BC$, it muſt be therefore that $CD \times AB + AD \times BC = BD \times AC$, two things equal to a third, being equal one to another. Q. E. D.

---

* *Sup. n.* 18. † *Sup. n.* 19. ‡ *B.* 3. *n.* 56.
§ *B.* 2. *n.* 43. ‖ *Sup. n.* 10.

† *B.* 3. *n.* 56. ‡ *B.* 2. *n.* 124. § *B.* 2. *n.* 40.
‖ *B.* 2. *n.* 80.

* *Sup. n.* 8. † *Sup. n.* 10. ‡ *B.* 3. *n.* 56.
§ *B.* 3. *n.* 56. ‖ *B.* 3. *n.* 17.

THEOREM 9. IF *the Diameter of a Circle be divided, ſo that one of its parts be Quadruple the other, and there be raiſed upon the Point of diviſion a Perpendicular bounded by the Circumference, I ſay, that the Square of the Chord drawn from the End of the Diameter to the End of the Perpendicular, is equal to five times the Square of the leſſer Part of the Diameter.* 63

LET NM be the Diameter of the Circle X ſo divided in A, that $NA = 4AM$, if the Perpendicular AB be erected and there be drawn the Chords NB, and BM, forming the right angled Triangle NBM; * I ſay, that $\overline{MB}^{2} = 5\overline{AM}^{2}$. Let $AM = a$, then, according to the Suppoſition, $MN = 5a$, let $MB = b$, then ∺ $5a.\ b.\ a$, † therefore $bb = 5aa$; ‡ now let $BA = d$; hence ∺ $a.\ d.\ 4a$. § by the ſame Reaſon therefore $4aa = dd$; *which was to be proved.*

CORALLARY 1. THE *Diameter NM being divided into ever ſo many parts, the Square of MB will be equal to as many times that of each of the Parts, as there are parts.* 64

IF, *as above and ſame Figure.* NM is ſo divided at the Point A, that AM be the Fifteenth (*or any other part*) of NM, it will always be, ‖ ∺ MA. MB. MN, or ∺ $5\ a.$ MB. a; therefore the Square of MB is alſo equal to $5aa$. *

CORALLARY 2. THE *Square of BA is equal to ſo many times the Square of each of the Parts, are there are parts leſs one.* 65

FOR *fig. preceed.* Suppoſing always the ſame, or any other Diviſion whatſoever, the Proportion will continue to be the Same, ∺AM. BA. AN, or ∺ a, BA. $4a$. † then $4aa$ or $5aa - aa$ is equal to the Square of AB, ‡ that the Square of AB is equal to ſo many times the Square of each of the Parts, as there are parts leſs one, is *what was to be proved.*

THEOR. 10. IF *a Line be divided into mean and extreme Ratio; I ſay that the Square of half the Line and its greateſt Segment, is equal to five times the Square of the ſaid half Line.* Eucl. 13. Prop. 1. 66

LET BA be divided into mean and extreme Ratio at the Point C, and let $AB = 2a$, and $BC = b$, then $CA = 2a - b$, it muſt be demonſtrated that the Square of $a + b$ (*which is* $aa + 2ab + bb$) is equal to $5aa$, taking away aa from each part, there will remain only to prove that $2ab + bb$ is equal to $4aa$. By the Suppoſition $2a.\ b :: b.\ 2a - b$; § then $bb = 4aa - 2ab$, ‖ and adding $2ab$ to each part, it will be $bb + 2ab = 4aa$; * which remained to be proved; then ſubſtituting aa which was before taken away, it will be $aa+bb+2ab=5aa$; *Q. E. D.*

---

* *B.* 2. *n.* 44. † *Sup. n.* 28. ‡ *B.* 3. *n.* 57.
§ *Sup. n.* 28. ‖ *Sup. n.* 28.

* *B.* 3. *n.* 57. † *Sup. n.* 28. ‡ *B.* 3. *n.* 17.
§ *Sup. n.* 34. ‖ *B.* 3. *n.* 57.

* *B.* 3. *n.* 6.

67 THEOREM 11. Two *right Lines being divided into mean and extreme Ratio, they are ſimilarly divided.* Eucl. 14. Prop. 2.

LET the two Lines be Z and X divided into mean and extreme Ratio at the Points C and G. Hence Z. AC. AC. CB, and X. EG: EG. GF. Let AC$=a$, and CB$=c$, and EG$=m$, and GF$=n$, it muſt be demonſtrated that AC. CB::BG. GF, or that $a. c::m. n$. As the Square of half Z together with a, is equal to five times the Square of half Z; ſo likewiſe the Square of half X together with m, is equal to five times that of half X, hence theſe Squares being Proportional, their Sides are alſo Proportional * ſeeing they are in Subduple Ratio, wherefore half Z$+a$. half Z:: half X$+m$. half X: and *dividedly* † a. half Z::m. half X, the Conſequents being changed and doubled ‡ Z. a:: X. m, but by the Suppoſition ÷ Z. a. c. and ÷ X. m. n. therefore the two Lines are divided Proportionally; § *Q. E. D.*

68 THEOR. 12. A *Line being divided into two unequal parts, if the Quintuple of the Square of the half thereof is equal to the Square of that half together with its longer part, the Line is then divided into mean and extreme Ratio, whoſe mean part is the greater Segment* Eucl. 13. Prop. 2.

LET AB be a Line unequally divided into two parts at the Point C, let AB$=2a$, and AC$=b$, hence CB$=2a-b$. It muſt be demonſtrated that if five times the Square of a, the half of AB, is equal to the Square of $a+b$, that is, to the Square of half a together with b, which I expreſs thus $5aa=aa+2ab+bb$, then the Line AB is divided into mean and extreme Ratio, and AC or b, will be its longer part, to prove this it muſt be ſhewn that if AB is ſo divided, then $5aa=aa+2ab+bb$. Let us ſuppoſe then that ÷ $2a$. b. $2a-b$; then $4aa-2ab=bb$; and by adding $+2ab$ to each part, it will be $4aa=2ab+bb$, ‖ alſo by adding aa to each part, it will be $5aa=aa+2ab+bb$; *which was to be proved.*

69 THEOR. 13. IF *a right Line is divided into mean and extreme Ratio, the Square of the leſſer Part together with half the greateſt, is Quintuple the Square of half the greater part.* Eucl. 13. Prop. 3.

LET AB be a Line cut into mean and extreme Ratio at the Point C, (*fig. preceed.*) Let AC$=2m$, and CB$=n$, then AB$=2m+n$. It muſt be demonſtrated that the Square of half the greateſt Part AC, together with the leſſer part BC, that is, the Square of $m+n$, (*which is* $mm+2mn+nn$) is Quintuple the Square of half AC, that is, of m, which is mm, according to the Suppoſition ÷ $2m+n$. $2m$. n; then $2mn+nn=4mm$; * adding mm,

---

* *B.* 3. *n.* 84. † *B.* 3. *n.* 51. ‡ *B.* 3. *n.* 45.
§ *B.* 3. *n.* 53. ‖ *B.* 3. *n.* 55.
* *B.* 3. *n.* 57.

mm, to each part it will be $mm+2mn+nn=5mm$; Q. E. D.

THEOR. 14. IF *a right Line is divided into mean and extreme* 70
Ratio, the Square of the whole Line, together with that of its leſſer part, is triple the Square of the greater part. Eucl. 15. Prop. 4.

LET AB be divided into mean and extreme Ratio, at the Point C, (*fig. preceed.*) Let $AB=Z$, and $AC=a$; and $BC=e$; then ∺ $Z.\ a.\ e.$ It muſt be demonſtrated that $ZZ+ee=3aa$. 1. $ZZ=aa+2ae+ee$. * It muſt therefore be demonſtrated that $aa+2ae+2ee=3aa$. 2. $ae+ee=Ze$, † and $Ze=aa$, ‡ by the Suppoſition, hence $ae+ee=aa$, there may then in order to obtain a Proof, be ſubſtituted in the Equation in the Room of $2ae+2ee$, their equal value $2aa$, and it will be $aa+2aa=3aa$; *which was to be proved.*

THEOR. 15. SIMILAR *Triangles and Parallelograms, are one* 71
to another in Ratio duple that of their Sides, which are of the ſame Ratio. Eucl. 6. Prop. 19.

LET ABD and EFG be two ſimilar Triangles, let fall the Perpendiculars AC and GH; the Triangles EGH, and BAC, being right Angled, and the Angle B being equal to the Angle E, they are Similar; then BC. AC::EH. GH. § alſo the two Triangles DAC and FGH being Similar; then likewiſe CD. AC::FH. GH, hence BC+CD. AC::EH+HF. GH. ‖ Let $AC=b$, and $BD=d$, $GH=m$, and $EF=n$, hence $b.\ d::m.\ n$; and *Alternately*, $b.\ m::d.\ n$. * The Surface of ABD is the half of bd, † and that of GEF is the half of mn; but the Ratio of bd to mn, is compounded of the two Ratio's of b to m, and of d to n; ‡ now theſe two Ratio's are the ſame: Therefore, according to the Definition of a duple Ratio, § the Ratio of bd to mn, is duple. Let BDMN and EFKI be two ſimilar Parallelograms, by the ſame Reaſoning it will be proved that $b.\ d::m.\ n$, and $b.\ m::d.\ n$, the Surface of BDMN is bd, and that of EFKI is mn; the Ratio of bd to mn, is compoſed of the two Ratio's of b to m, and of d to n; theſe two Ratio's are equal; the Ratio compounded thereof is therefore duple.

LEMMA. SIMILAR *Polygons may be divided into an equal num-* 72
ber of Triangles, ſimilar each to each.

LET the two ſimilar Polygons be ABCDE, and FGHIK; I ſay that they may be divided into an equal number of Triangles, each Similar to its correſponding one; for 1. Seeing the Polygons are Similar, they have each an equal number of Sides, and the Angles which they contain are equal by the Definition; ‖ hence from one of the equal Angles, as from E and K, may be drawn Lines to the other Angles which will divide the Polygons into an equal number of Triangles. * 2. All the Triangles of one Polygon are Similar to thoſe of the other, each to its correſponding one; thus the Triangle ABE is Similar to its correſponding one FGK; for by the Suppoſition the Angle EAB=KFG; and their containing Sides are Proportional; that is, EA. AB::KF. FG; and therefore the two Triangles are Similar. † It is the ſame of the other Triangles, whoſe Angles contained between homologous Sides are equal, whether they be form'd by the Sides of the Polygons, or by the remaining ones after the equal ones which touch at the Points E and K are taken away, ſuch as BEC and GKH, therefore &c. *which was to be proved.*

---

* *B.* 3. *n.* 20. † *B.* 3. *n.* 19. ‡ *B.* 3. *n.* 57.
§ *Sup. n.* 10. ‖ B. 3. *n.* 63.

* *B.* 3. *n.* 46. † *B.* 2. *n.* 135. ‡ *B.* 3. *n.* 77.
§ *B.* 3. *n.* 72. ‖ *Sup. n.* 38.

73 THEOREM 16. ALL *ſimilar rectilinear Figures, are one to another as the Squares of their homologous Sides.*

LET the ſame things and the ſame Figure be ſuppoſed as in the preceeding Lemma. 1. Each of the Figures may be divided into an equal number of Triangles ſimilar one to another, each to each, by the preceeding Lemma, &c. 2. Each of the Triangles of one Figure will be to each of its correſponding ones in the other Figure, in duple Ratio of each of the homologous Sides, by the preceeding Theorem, ‡ that is, in duple Ratio of AE to FK, or AB to FG, but the Square of AE is to that of FK, in duple Ratio of AE to FK; § and it is the ſame of all the other Triangles which compoſe the two Figures, whoſe Sides preſerve the ſame Proportion by reaſon of their likeneſs, wherefore as each Triangle of the one is to its like in the other, ſo are all the Triangles of one to all the Triangles of the other, ‖ *that is*, the firſt Figure is to the Second, as the Square of the Side AE is to that of FK; and it is the ſame of all the Reſt pitch upon which you will, therefore. &c. *which was to be concluded.*

74 CORALLARY. IF *four Lines are Proportional, ſimilar Figures deſcribed upon thoſe Lines are Proportional, and if the ſimilar Figures are Proportional, the four Lines are Proportional.* Eucl. 6. Prop. 22.

LET the four proportional Lines be *a. b*::*c. d*, upon which let there be deſcribed four ſimilar Figures V. X. Y. Z. by the preceeding Theorem V. X::*aa. bb*, and Y. Z::*cc. dd*, but *aa. bb*::*cc. dd*; * therefore V. X::Y. Z, and if this be ſo, then *a. b*::*c. d*; for the equal duple Ratio's are compounded of equal Ratio's, therefore if *aa. bb*::*cc. dd*, it muſt be that *a. b*::*c. d*.

75 THEOREM 17. TRIANGLES *and Parallelograms having the ſame Height, are one to another in the ſame Ratio as their Baſes.*

LET Z and X, be either two Parallelograms or two Triangles, having the ſame Height which I call *a*, the Baſe of Z is *b*, and that of X is *d*, the Surface of Z is *ab*, if it is a Parallelogram;

---

* *B.* 2. *n.* 122. † *Sup. n.* 13. ‡ *Sup. n.* 71.
§ *B.* 3. *n.* 80. ‖ *B.* 3. *n.* 50.
* *B.* 3. *n.* 84.

gram; but only the Half thereof, if it is a Triangle: * *ad* is alſo the Surface of X; or if it is a Triangle, then only the Half thereof; now *ab. ad*::*b. d*; † therefore Z and X are one to another, as their Baſes.

Problem 1. A *Point being given in one of the Sides of a Triangle, to draw a Line thro' that Point ſo as to divide the Triangle according to a given Ratio.* 76

Let ABC be a Triangle required to be divided according to a given Ratio, by drawing a right Line thro' the Point D. Let this Ratio be that of BG to GC; if D and G were one and the ſame Point, there ſhould be drawn a Line from D to A, for BAD and DAC are one to another as BD and DC; ‡ therefore what was required is done. If G and D are two different Points, draw GH parallel to AD, and from D to H, where that Parallel cuts AC, draw another Line *viz.* DH; it muſt be proved that it divides the Triangle ABC, according to the Ratio of BG to GC. The two Triangles BAG and GAC are one to another as BG to GC, § it is then required only to prove that $DHC=AGC$, and $DBAH=BAG$; hence as DHC. AGC::DBAH. BAG; the two Triangles GAH and GHD upon the ſame Baſe GH, and between the ſame Parallels are equal; ‖ then $GHC+CDH$, or $DHC=GHC+GAH$, or GAC, it remains to prove that $ABDH=ABG$; which is eaſy; for by the ſame Reaſon as before mentioned, the Triangle ADG is equal to that of AHD; therefore adding to each the Triangle ABD, it will be $ABDH=ABG$; and therefore ABDH. DHC::ABG. ABC, therefore that is done which was to be done.

Problem 2. To *divide a Parallelogram according to a given Ratio, by drawing a Line thro' a given Point thereof.* 77

Let the Parallelogram be ABCD, it is required to be divided according to the Ratio of BF to FC, by drawing a Line thro' the given Point E. 1. Draw FG Parallel to AB, or to CD, the two Parts ABFG and CDGF are one to another, as BF is to FC, according to the laſt Theorem, * 2. Take GH equal to FE, and having drawn a right Line from E to H, I ſay that ABEH and ECDH are the Parts required: For HGI and EFI are two ſimilar Triangles; ſeeing they are Equiangular: For the Angles at the Point I are equal, † and the other Angles are likewiſe equal, being alternate ones between the ſame Parallels; ‡ therefore the Triangles having the Sides EF and GH equal, are entirely equal. § Therefore $ABFIH=ABFG-HGI$; and there being added to each the Value of HGI equal to FIE, it will be $ABFIH+FEI=ABFG$; which was to be demonſtrated. It is proved the ſame Way that $CDHE=CDGF$.

Theorem

---

* *B.* 2. *n.* 133. † *B.* 3. *n.* 54. ‡ *Sup. n.* 75. § *Sup. n.* 75. ‖ *B.* 2. *n,* 134.

* *Sup. n.* 75. † *B.* 2. *n.* 21. ‡ *B. n. n.* 25. § *B.* 2. *n.* 96.

78 THEOREM 18. THE *Square of the Hypothenuſe of a right angled Triangle is equal to the Sum of the Squares of the two other Sides.* Eucl. 1. Prop. 47.

A *Demonſtration different from that laid down in Book* 2. *n.* 142.

ABC is a right Angled Triangle, the right Angle is A, and the Hypothenuſe is BC. It muſt be proved that $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$; and that differently from what has been done already, * From the right Angle A, let fall the Perpendicular AD and prolong it, then according to what has been proved. † $\div$ BC. BA. BD; then ‡ $\overline{AB}^2 = BC \times BD$, or $BE \times BD$ ſince that $BC = BE$, becauſe it is a Square: So likewiſe $\div$ BC. AC. CD; therefore $\overline{AC}^2 = BC \times CD$, or $CF \times CD$ ſeeing that $CF = CB$; now $\overline{BC}^2 = BE \times BD + CF \times CD$: § Therefore $\overline{BC}^2 = \overline{AB}^2 + \overline{AC}^2$; *which was to be*
79 *proved.*

CORALLARY 1. WHAT *Figure ſoever, is made upon the Hypothenuſe of a right angled Triangle, it is equal to the ſimilar Figures deſcribed and in a like Poſition upon the other two Sides thereof.* Eucl. 6. Prop. 31.

UPON the Sides BD. DA. AB, let there be three ſimilar Figures X. Y. Z, let X be upon the Hypothenuſe, theſe Figures are one to another as $\overline{BD}^2$, $\overline{AD}^2$, $\overline{AB}^2$. ‖ now $\overline{BD}^2 = \overline{AD}^2 + \overline{AB}^2$ by
80 the Theorem; therefore $X = Y + Z$.

CORALLARY 2. A *Perpendicular being drawn from the right Angle of a rectangled Triangle upon the Hypothenuſe, the Squares deſcribed upon the other two Sides are one to another as the Parts or Segments of the Hypothenuſe.*

LET ABD be a rectangled Triangle, whoſe Hypothenuſe is DB, upon which the Perpendicular AC falls from the right Angle A. It muſt be proved that $\overline{AD}^2 . \overline{AB}^2 :: DC . CB$, ſince that $\div$ DB. DA. DC; * then $DB \times DC = \overline{DA}^2$; † by the ſame Reaſon $\div$ DB. BA. BC; then alſo $DB \times CB = \overline{BA}^2$, wherefore $DB \times DC . DB \times BC :: \overline{AD}^2 . \overline{AB}^2$; but $DB \times DC . DB \times BC :: DC . CB$;
81 ‡ hence, $\overline{AD}^2 . \overline{AB}^2 :: DC . CB$, *Q. E. D.*

THEOREM 19. PARALLELOGRAMS *having one Angle common or equal, are in a compound Ratio of the containing Sides.* Eucl. 6. Prop. 23.

---

* *B.* 2. *n.* 142. † *Sup. n.* 28. ‡ *B.* 3. *n.* 57. § *B.* 3. *n.* 18. ‖ *Sup. n.* 73.

* *Sup. n.* 28. † *B.* 3. *n.* 57. ‡ *B.* 3. *n.* 54.

THE Angle ABC is equal to the Angle GHI, I ſay that theſe two Parallelograms are in Ratio compounded of the Ratio's of AB to GH, and of BC to HI, the Parallelograms ABCE and GHIK are equal to the Rectangles BDFC and HLMI each to each, * which are in Ratio compounded of the Ratio's of BD to HL, and of BC to HI, † now the two Triangles ABD and GHL being Similar, the Ratio of DB to HL is the ſame as that of AB to GH, for. 1. They are Rectangled. 2. The Angles ABC and GHI being equal, if the right Angles DBC and LHI be taken away, the Remainders ABD and GHL are equal; hence they are Equiangular, ‡ and therefore their Sides are Proportional. §

CORALLARY 1. WHEN *two equal Parallelograms have one Angle common or equal, the containing Sides thereof are in reciprocal Proportion.* Eucl. 6. Prop. 14. 82

IF the Parallelograms ABCE and GHIK are equal one to another, *fig. above,* I ſay that AB. BC. GH. HI, are in reciprocal Proportion; *that is,* that AB. GH::HI. BC. Seeing that $BD\times BC = HL\times HI$, then ‖ BD. HL::HI. BC, now ſince that the Angles ABC and GHI are equal by the Suppoſition, the Concluſion will be * that AB. GH::BD. HL::HI. BC; and therefore AB. GH::HI. BC; † *Q. E. D.*

CORALLARY 2. IF *the Surfaces of two Triangles are equal, and they have one Angle common or equal, the Sides containing that Angle are in reciprocal Proportion.* Eucl. 6. Prop. 15. (*ſame figure*) 83

LET the two Triangles be ABC and GHI whoſe Surfaces are equal, and the Angles ABC and GAI equal, it muſt be demonſtrated that AB. GH::HI. BC, theſe Triangles are half the Parallelograms ABCD and GHIK, which proves that AB. GH::HI. BC; *which was to be proved.*

PROB. 2. TO *find a Square equal to a given Rectangle.* 84

LET bd be the given Rectangle ſuppoſe x to be a mean Proportional between b and d, ‡ then $\div\div\ b.\ x.\ d$, and $bd = xx$, § thus is a Square found equal to a given Rectangle.

PROB. 4. TO *deſcribe upon a given right Line, a right lined Figure ſimilar to a given rectilinear Figure, and in a like Poſition thereto.* Eucl. 6. Prop. 28. 85

LET AB be a right Line, upon which muſt be deſcribed a ſimilar Figure and in a like Poſition to the Rectilinear one CDFE, reduce CDFE into two Triangles, ‖ upon AB make the Triangle ABH, ſimilar to the Triangle CDF, ¶ and upon AH the Triangle AGH ſimilar to CEF, and ſo on, if the Figure had con-

---

* *B.* 2. *n.* 129. † *B.* 3. *n.* 77. ‡ *B.* 2. *n.* 80. § *Sup. n.* 10. ‖ *B.* 3. *n.* 58.

* *Sup. n.* 7. † *B.* 3. *n.* 53. ‡ *Sup. n.* 89. § *B.* 3. *n.* 57. ‖ *B.* 2. *n.* 122.

¶ *B.* 2. *n.* 97.

tained more Triangles. the Sides of theſe ſimilar Triangles are Proportional, † but as they form or compoſe the two Figures, their Angles are therefore alſo equal, and their homologous Sides proportional, thus AB. AG::DC. CE, and conſequently the Figures are Similar; ‡ *which was to be proved.*

86 Prob. 5. Two *rectilinear Figures being given, to deſcribe a third Similar to one of them and equal to the other.* Eucl. 6. Prop. 25.

The given rectilinear Figure is A, 'tis required to make another Similar thereto and equal to B, another given right lined Figure. Upon CD deſcribe the Parallelogram CE equal to A, § and likewiſe upon DE the Parallelogram DH equal to B having the given Angle D, then find IK a mean Proportional between CD and DG, ‖ upon IK, deſcribe L ſimilar to A; ¶ I ſay that L is the Figure required, it muſt be proved that L is equal to B. By conſtruction A and L are Similar to each other, therefore they are one to another as $\overline{CD}^2$ is to $\overline{IK}^2$, ¶ that is, in duple Ratio of CD to IK; ⫲ but by the Hypotheſis ∺ CD. IK. DG; hence CD is alſo to DG in duple Ratio of CD to IK. † and conſequently CD. DG::A. L, but alſo CD. DG::CE. DH. ‡ and by conſtruction CE=A, and DH=B; then CD. $DG::\left\{\begin{array}{l}A.\ B.\\ \hline A.\ L.\end{array}\right.$ wherefore B=L; § *Q. E. D.*

Observe *that this Conſtruction would have been eaſier, if inſtead of the indifferent Parallelogram CE, there had been made a Rectangle.*

87 Theor. 20. A *Square whoſe Circumference is equal to that of a right angled Parallelogram, is greater than the Parallelogram by the Square of half the Difference, which is between the Sides of the Parallelogram.*

The Square is X, and Z a Parallelogram having an equal Circumference, the Sum of the two Sides either of X or Z is AC, the half of AC, which is AB, is the Side of the Square X. The longeſt Side of Z is AE, and the Shorteſt EC. Let AB or BC $=b$, and BE $=a$, and that BD be taken equal to a, then AE the longeſt Side of Z will be $b+a$, and the ſhorteſt EC or AD will be $b-a$, their difference will then be $2a$, the Square X is therefore equal to bb, and the Rectangle Z equal to the Product of $b+a\times b-a$, that is, to $bb-ab+ab-aa$ or $bb-aa$; conſequently the difference between X and Z is aa, the Square of half the difference of the Sides; *which was to be proved.*

Theorem

---

† *Sup. n.* 10. ‡ *Sup. n.* 38. § *B.* 2. *n.* 139 *and* 140.
‖ *Sup. n.* 29. ¶ *Sup. n.* 85.
¶ *Sup. n.* 73. ⫲ *B.* 3. *n.* 80. † *B.* 3. *n.* 72.
‡ *Sup. n.* 75. § *B.* 3. *n.* 52.

THEOR. 21. OF *two regular Polygons having the ſame Circumference, that which hath moſt Sides hath the greateſt Surface.* 88

A Polygon is equal to a Triangle whoſe Baſe is the Circuit of the Polygon, and its Perpendicular height the Apotome thereof. * Let there be therefore two Polygons of the ſame Circuit; the Half of which I call x, and let b be the Apotome of that which hath moſt Sides, and let that of the other be d; then their Surfaces are xb and xd; † Now $xb.\ xd::b.\ d$, ‡ and $b>d$; § therefore xb is greater than xd.

CORALLARY. THE *Circle is the moſt Capacious of all Iſoperimetrial Figures, that is, having the ſame Circumference.* 89

FOR a Square is greater than a Parallelogram of the ſame Circuit, ‖ the Square is a Polygon, and conſequently the Circle conſidered as a Polygon of an infinite number of Sides, has the greateſt Apotome, therefore a greater Surface. It is the ſame of all other Polygons.

THEOR. 22. REGULAR *and ſimilar Polygons are in Ratio duple that of their Sides.* Eucl. 6. Prop, 20 90

LET the two ſimilar Polygons be X and Z; they are equal to two ſimilar Triangles, * and if a is half the Circuit of X, and b its Apotome, and c half the Circuit of Z, and d its Apotome, $ab=X$. and $cd=Z$; now ab is to cd in Ratio duple that of a to c, or that of b to d, † which by the Hypotheſis is equal thereto, the Figures being Similar.

CORALLARY. THEREFORE *the Surface of one Circle is to that of another, in a duple Ratio of its Circumference, or of its Diameter.* 91

THEOR. 23. SIMILAR *and regular Polygons, are one to another as the Squares of the Diameters of their circumſcribing Circles.* Eucl. 12. Prop. 1. 92

LET the two inſcribed Polygons be X and Z. It muſt be demonſtrated that they are one to another as the Squares of their Diameters * AD, GK. Seeing that X and Z are ſimilar Polygons, they are one to another in duple Ratio of the Side AB to the Side GH, by the preceeeding Theorem; but on account of their Similiarity, the Triangles ABD, GHK are alſo Similar; for their Angles are all equal, being placed upon ſimilar Arcs, ‡ the homologous Sides are therefore Proportional, § and therefore AB. GH::AD. GK; hence the duple Ratio of AB to GH is the ſame as the duple Ratio of AD to GK, therefore the Polygon X is to the Polygon Z, in duple Ratio of the Diameter AD to GK, *that is*, as the Square of AD to the Square of GK; ‖ *Q. E. D.*

CORALLARY. THEREFORE *ſeeing that Circles may be taken for Polygons, their Surfaces are one to another as the Squares of their Diameters.* Eucl. 12. Prop. 2. 93

---

* B. 2. *n.* 145. † *B.* 2. *n.* 135. ‡ *B.* 3. *n.* 54. § *Sup. n.* 48. ‖ *Sup. n.* 87.

* *B.* 2. *n.* 145. † *Sup. n.* 71. ‡ *B.* 2. *n.* 39. § *Sup. n.* 10. ‖ *B.* 3. *n.* 80.

94 **THEOREM 24.** $AB=BC$; *the Parallelogram ABDH is greater than the Rectangle AFGK, and than any other whoſe point G is in the Diagonal DC.* Eucl. 6. Prop. 27.

$BG=GE$, * then $BG+GC=GE+GC$, *that is*, that $BI=KE$. Now $BF=BI$, ſince that $AB=BC$; then $BF+BG$, or $AG=BG+KE$; now $BG+KE$ is leſſer than BE, or its equal AD: therefore AG is leſſer than AD.

95 **PROB. 6.** To *apply to a given Line AB, a Parallelogram equal to C, wanting a Parallelogram ſimilar to D. C ought not to be greater than a Parallelogram made Similar to D, and applied to the half of AB, that is, greater than AF or EG.* Eucl. 6. Prop. 28.

DIVIDE AB equally at the Point E, upon EB deſcribe the Parallelogram EG ſimilar to D, † which according to what is required; will be greater than C, let the Figure I be the Exceſs thereof, let the Parallelogram NT be made equal to I, and ſimilar to D or EG by the Problem, ‡ let there be drawn the Diagonal FB, and make $FO=KN$, and $FQ=KT$, thro' O and Q draw the Parallels SR and QZ: then is AP the Parallelogram required. For the Parallelograms D, EG, OQ, NT, ZR are Similar one to another, and $EG=NT+C=OQ+C$; conſequently C is equal to the Gnomon $OPQ=AO+PG=AO+EP=AP$; *Q. E. D.*

96 **PROB. 7.** To *apply to a given right Line AB, a Parallelogram equal to C, and exceeding it by a Parallelogram ſimilar to D.* Eucl. 6. Prop. 29.

LET AB be equally divided in E, upon EB deſcribe EG ſimilar to D; § then by means of the former Problems, ‖ deſcribe HK equal to $EG+C$ and ſimilar to D or to EG, prolong FE and FG, ſo that $FL=IH$ and $FM=IK$; thro' L and M draw the Parallels RN, MN and AR; prolong AB and GB; draw the Diameter FBN, and then AN is the Parallelogram required. For theſe four Parallelograms D, HK, LM, EG are Similar *by conſtruction*; therefore OP is ſimilar to LM, or to D. * So likewiſe $LM=HK$, *by conſtruction*; hence $LM=HK=EG+C$; then, $LM=EG+C$; then $LM-EG=C$. Now $LM-EG$ is equal to $BM+LB+BN$, or equal to $BL+AL+BN$; for $AE=EB$ *by the Hypotheſis*; therefore $C=BL+AL+BN$, or $C=AN$; *Q. E. D.*

PROB. 8. A *right lined Figure being given, to find another Similar thereto and in a given Ratio therewith.*

97 LET the given Figure be A, 'tis required to find X, to which the Surface A hath a given Ratio, as for Example of 5 to 1. Theſe Figures being ſuppoſed Similar, they ought to be one to

---

* *B.* 2. *n.* 132. † *Sup. n.* 85. ‡ *Sup. n.* 86.

§ *Sup. n.* 85. ‖ *B.* 2. *n.* 139. and *Sup. n.* 86.

* *Sup. n.* 51.

another as the Squares of their homologous Sides, * *that is,* in duple Ratio of the ſaid Sides, † or the ſame of ſome other homologous Line, Chord or Diameter which joins the Angles as B and C, therefore to do this, find ſuch a mean Proportional C; as may be as 5 to 1; ‡ between the Diameter B and another Line D, ſo that ∺ B. C. D. and upon the Line C deſcribe the Figure X, ſimilar to A, § theſe two Figures A and X are one to another as the Squares of B and C, ‖ or in duple Ratio of B to C, *that is,* as to B to D, or 5 to 1; *which was to be done.*

98 Corallary. It *is therefore eaſy to find a Figure ſimilar to a given one, and leſs than another by a certain Figure given.*

By taking ſuch another Line D as may bear the required Proportion to the Line B, the Side of the given Figure, and between them find as has been taught, * a mean Proportional C, and upon the ſame as a Side deſcribe a Figure ſimilar to the given one, † which *by the Problem,* will be the Figure required.

99 Corallary 2. It *is equally eaſy by the preceeding Conſtruction, to find a Figure ſimilar to a given one, exceeding it by a certain Figure given.*

SECT. V.

Of the Commenſurability, or Incommenſurability of Lines and Surfaces.

Definition I.

100 Quantities *are called Commenſurable, when they can be meaſured by a Third, which is therefore their common meaſure.*

The common Meaſure of a Yard and an Ell, is an Inch, which is contained exactly thirty ſix times in a Yard, and forty five times in an Ell, meaning the Ell Engliſh.

101 Def. 2. Those *Quantities which cannot be meaſured by any common Meaſure, are Incommenſurable.*

A length of ſeven Ells cannot be exactly meaſured by a Yard; but it may be meaſured by a leſſer Meaſure, as by a Foot and by an Inch. Hence two Quantities are incommenſurable only when they cannot be exactly meaſured, by any meaſure how little ſoever it be.

102 Corallary. Therefore *two Quantities having an infinite Proportion, are incommenſurable to each other.*

---

* *Sup. n.* 73. † *B.* 3. *n.* 80. ‡ *Sup. n.* 29. § *Sup. n.* 95. ‖ *Sup. n.* 73.

* *Sup. n.* 29. † *Sup. n.* 8.

For if they were Commenſurable, their common meaſure wou'd determine their Proportion, there is an infinite Proportion between a Point and a Line; for it cannot be determined in any Line whatſoever, how many Points are therein, ſeeing that it may be divided *ad inf.* So likewiſe the Proportion between a Line and a Surface is infinite; for in a Surface may be conceived as many Lines you pleaſe, ſo you may of Surfaces in a Solid.

103 Def. 3. A *rational Quantity is that whoſe Proportion to another is known and determined, and which is expreſſible by Number.*

104 Def. 4. Two *Quantities which are not commenſurable in themſelves, are commenſurable in Power, if their Squares or their Cubes are commenſurable.*

If b and c are Incommenſurable, but that their Squares are Commenſurable, as for Example let bb, be to cc as 3 to 5, then b and c Incommenſurable in themſelves, are Commenſurable in the ſecond Power. If x and z are not Commenſurable nor their Squares xx and zz, but only their Cubes xxx and zzz, or x^3, z^3, for Example that if x^3 be to z^3 as 10 to 13, then x and z Incommenſurable in themſelves and in the ſecond Power, are Commenſurable in the third power.

105 Def. 5. A *ſquare Number, is the Product of a Number multiplied by itſelf.*

Thus 9 is a ſquare Number, becauſe 'tis the Product of 3 multiplied by 3, which is the Root thereof.

106 Def. 6. A *cube Number is the Product of a ſquare Number multiplied by the Root of the Square.*

Thus 8 is cube Number made by the ſquare Number 4 multiplied by the ſquare Root 2, *that is*, by 2, which multiplied by itſelf makes the Square 4.

107 Def. 7. Those *Numbers whoſe Square Roots are not expreſſible by Number, are not ſquare Numbers.*

108 Def. 8. Those *Numbers whoſe cube Roots cannot be expreſſed by Number, are not cube Numbers.*

2, 3, 5, 6, 7, 8, 10, are Numbers which have no ſquare Root expreſſible by number, for no number can be found which multiplied by itſelf makes either 2, 3, 5, 6, 7, 8, or 10, &c. Nor is there any cube Number among them except the 8, for there is not any number, which Cubically multiplied, can produce them.

109 Def. 9. The *exponential Numbers of a Ratio, are the leaſt Numbers which expreſs it.*

Thus if x is to z, as 6 to 12, the Exponents of the Ratio of x to z are 1 and 2, which are the leaſt Numbers that can be between them, as 6 and 12.

When two incommenſurable Quantities are joined together, and which conſequently cannot have the ſame Name, or which muſt be expreſſed by two different Names, this is called a *Binomial.*

When from one Quantity another which is Incommenſurable thereto is ſubtracted, and that it cannot therefore be expreſſed with only one Sign, 'tis alſo a *Binomial*. But to diſtinguiſh this from that which is made by the Sum of two incommenſurable Quantities, 'tis called an *Apotome*, *Reſidual* or diminiſhed Quantity.

Evident Propoſitions.

PROPOSITION I.

TWO *Numbers are always commenſurable one to another, for they have Unit (at leaſt) for their common Meaſure.* 110

FOR Example, in theſe two Numbers 77 and 81, as in all others, the Unit is contained exactly a certain Number of Times, therefore it is their common meaſure.

PROP. 2. WHEN *one Number is ſubtracted from another, the Remainder is a Number.* 111

FOR the Remainder is one or more Units.

PROP. 3. LINES *and Surfaces which are as Number to Number, are commenſurable; and thoſe which are Incommenſurable are not as Number to Number.* 112

PROP. 4. TWO *Quantities commenſurable to a Third, are commenſurable to each other.* 113

FOR if B and Z are commenſurable, their Ratio may be expreſſed by Number. If C and Z are likewiſe commenſurable, their Ratio is alſo expreſſible by number; therefore theſe three Quantities may be all denoted by number; they are therefore commenſurable.

PROP. 5. A *Line, root or ſide of a Square which is not a Square number, is Irrational, if its Square were equal to a Square Number, it would be Rational.* 114

THE Root of an irrational Number cannot be expreſſed by any Number whatſoever; therefore every Line equal to that Root, is inexpreſſible by number.

PROP. 6. IF *the Squares of two Lines are not one to another as two ſquare Numbers, the two Lines are incommenſurable.* 115

THE Lines are each equal to the Root of an irrational Number, therefore inexpreſſible by Number, they are then incommenſurable.

PROP. 7. A *rational Square cannot be equal to two Squares, one of which is rational and the other Irrational.* 116

THIS is what was ſaid Propoſition the Second;* that in taking one Number from another, the Remainder is a Number. Thus if from a rational Square, *that is*, which is a Number, be taken a rational Square or a Number, the Remainder ought to be a rational Square or Number.

---

* *Sup. n.* 111.

117 **Prop. 8.** A *commenſurable Line being divided into two Parts, if one is commenſurable, the other is ſo likewiſe.*

For this Line is expreſſible by number, from which having taken away a commenſurable part, that is, a Number, the Remainder will be a Number, according to the ſecond Propoſition.

118 **Prop. 9.** If *two commenſurable Lines are added together, their Sum will be Commenſurable.*

This is evident, a Number added to a Number makes a Number.

119 **Prop. 10.** If *from a commenſurable Line be ſubtracted an incommenſurable one, the Remainder is Incommenſurable.*

For if one was Commenſurable, the other would be ſo likewiſe, according to the eighth Propoſition.

120 **Prop. 11.** Four *Lines being in Proportion, if the Firſt is commenſurable to the Second, the Third will be ſo to the Fourth. If the Firſt is Incommenſurable to the Second, the Third will be ſo to the Fourth.*

This is as much as to ſay, that if the Ratio of the Firſt to the Second is expreſſible by number; that of the Third to the Fourth, which is the ſame, is alſo expreſſible by Number. If the firſt Ratio is inexpreſſible, it is the ſame of the Second.

121 **Prop. 12.** The *Square of a rational Line, is a Square Number; its Cube is alſo a Cube Number.*

122 **Theorem 1.** If *ſimple Ratio's are of Number to Number, the Exponents of the duple Ratio's which are compounded thereof, are ſquare Numbers, and thoſe of the triple Ratio's Cube Numbers.*

Let the ſimple Ratio be that of b to d; then the Ratio of b^2 to d^2, is Duple, now b and d being expreſſible by Number their Squares are ſo likewiſe. Hence b^2 and d^2 are equal to ſquare Numbers, therefore b^2 and d^2, are one to another as ſquare Numbers. By the ſame Reaſon b^3 and d^3 are one to another as Cube Numbers

123 **Theor. 2.** A *ſimple Ratio is Surd, if its duple or triple Ratio be Surd.*

For *by the preceeding Theorem, if the Ratio was not Surd, the Exponent of the duple Ratio would be a Square Number, and that of the triple Ratio a Cube Number.*

124 **Theor. 3.** A *ſimple Ratio is not Surd, if the Exponential Numbers of the duple Ratio are Squares, and the Exponents of the triple Ratio Cubes, and if they be not ſo, it is a ſurd Ratio.*

For the Squares are in duple Ratio of their Sides or Roots, and the Cube Numbers triple of their Root; * hence the Roots of the Exponents of a duple or triple Ratio, are the Exponents of the ſimple Ratio; which conſequently is not ſurd, by the ſame Reaſon if the Numbers are neither Squares nor

---

* *B. 2. n. 73 and 75.*

Cubes; as their Roots are the Exponents of ſimple Ratio's, theſe ſimple Ratio's not being expreſſible by Number, they are ſurd Ratio's.

LEMMA. FOUR *Lines being in Proportion, the Product of the Antecedents is to that of the Conſequents, as the Square of the firſt Term is to the Square of the ſecond Term.* 125

LET them be $a.\ b :: c.\ d$, it muſt be proved that $ac.\ bd :: aa.\ bb$. the Product bc of the Means divided by the firſt Term is equal to the Fourth; * thus $\frac{bc}{a}$ may be put for d; therefore $\frac{bbc}{a} = bd$: Hence it muſt be demonſtrated that $ac.\ \frac{bbc}{a} :: aa.\ bb$. Multiply ac and $\frac{bbc}{a}$ by a, the Product is aac and bbc, and divide theſe products by c, there will remain aa and bb, the ſame Ratio will always remain; † therefore $ac.\ bd :: aa.\ bb$, *which was to be proved.*

THEOR. 4. FOUR *commenſurable Lines being in proportion, the Rectangle or Product of the Antecedents, is to that of the Conſequents, as two ſquare numbers.* 126

LET the four commenſurable and proportional Lines be $a.\ b :: c.\ d$; *by the preceeding Lemma,* $ac.\ bd :: aa.\ bb$. Now a and b are ſuppoſed Commenſurable; their Squares are therefore two ſquare Numbers, therefore ac and bd are one to another, as two ſquare Numbers.

LEMMA. SIX *Lines being Proportional, the Product of the Antecedents is to that of the Conſequents, as the Cube of the firſt Term is to the Cube of the ſecond Term.* 127

LET them be $a.\ b :: c.\ d :: e.\ f$. It muſt be proved that $ace.\ bdf :: aaa : bbb$; as $\frac{bc}{a} = d$, and $\frac{bc}{a} = f$: ‡ Then $bdf = \frac{b^3ce}{aa}$. Now in multiplying ace and $\frac{bbbce}{aa}$ by aa, and dividing the Products of that multiplication $aaace$ and $bbbce$ by ce, the Quotients will be a^3, b^3, the ſame Ratio will always remain; § therefore $ace.\ bdf :: a^3.\ b^3$. *which was to be proved.*

THEOR. 5. SIX *commenſurable Lines being in proportion, the Product of the Antecedents is to that of the Conſequents, as two cube Numbers.* 128

LET them be $a.\ b :: c.\ d :: e.\ f$; *by the preceeding Lemma,* $ace.\ bdf :: aaa.\ bbb$. Now a and b being Commenſurable, are therefore expreſſible by Numbers, ‖ a^3 and b^3 are two cube Numbers; conſequently ace and bdf are one to another as two Cube numbers.

---

* *B.* 3. *n.* 60. † *B.* 3. *n.* 54 *and* 55. ‡ *B.* 3. *n.* 60. § *B.* 3. *n.* 54 *and* 55. ‖ *Sup. n.* 103.

129 THEOR. 6. IF *three Lines are in continued Proportion, and that the Firſt is to the Third as two ſquare Numbers, the three Lines are Commenſurable.*

LET the continued Proportion be ∺ a. b. c, if the Ratio of a to c is a ſquare Number; I ſay that the Ratio of a to b will be a Ratio of Number to Number; for the Ratio of a to c is duple that of a to b, or as the Square of a to the Square of b; * and conſequently by *Theorem* 3. † this duple Ratio being as two ſquare Numbers, the ſimple Ratio of which it is compoſed is Rational.

130 CORALLARY. IN *this Caſe the Rectangle of the Extremes and the Sides are Commenſurable with the Square of the mean Term, and its Side.*

FOR 1. Seeing that by the Suppoſition ∺ a. b. c, it will follow that $ac=bb$; ‡ and therefore ac commenſurable with bb. 2. By the preſent Theorem, the Sides a. b. c, are commenſurable.

131 THEOR. 7. IF *three Lines are Proportional, and that the Firſt is to the Third as two Numbers, whoſe Product is not a Square Number (*or which have for Exponents two ſurd Numbers*) the mean Term will be Incommenſurable in itſelf, and Commenſurable in power with the Firſt and Third.*

LET ∺ b. c. d, the Firſt of the three Lines b is to the Third d, as theſe Numbers 2 and 9, whoſe Product 18 is not a ſquare Number, I ſay that the ſecond Line c is incommenſurable in itſelf, and Commenſurable in power with the Firſt and Third. It's Square cc is equal to bd, or to 18. § Now 18 not being a ſquare Number, its root c cannot be expreſſed by Number, therefore c is Incommenſurable in itſelf with b and d; but its power cc, which makes 18, is Commenſurable with b and d; *that is*, with 2 and with 9, ſeeing that is Ratio is expreſſible by number.

132 THEOR. 8. IF *of three Lines in continued Proportion, the Firſt is not to the Third as Number to Number, the mean Term is Incommenſurable with them, either in itſelf or in Power.*

LET the Proportion be ∺ b. c. d, whoſe firſt Term b is not to d as Number to Number; I ſay that neither c nor its power cc are Commenſurable with b and d; for the Ratio of b to d is duple that of b to c, and of c to d. ‖ this duple Ratio being therefore Surd, the ſimple Ratio of b to c, or of c to d is therefore Surd *by the ſecond Theorem.* ¶ The Square of b is to the Square of c as b is to d. † Then ſeeing that b is not to d as Number to Number, bb and cc are not as Number to Number; therefore they are Incommenſurable. ‡

---

* *B.* 3. *n.* 86. † *Sup. n.* 124. ‡ *B.* 3. *n.* 157.
§ *B.* 3. *n.* 57. ‖ *B.* 3. *n.* 83,
¶ *Sup. n.* 123. † *B.* 3. *n.* 84. ‡ *Sup. n.* 112.

THEOR. 9. IF *of four Lines in continued Proportion, the Ratio of the Firſt to the Fourth be a Ratio of Number to Number, having for Exponents Cube numbers, the four Quantities are Commenſurable.* 133

LET ∺ *b. c. d. f.* If *b. f*::1. 8, the four Lines *b, c, d, f,* are Commenſurable; for the Ratio of *b* to *f* is triple. ¶ The two Numbers 1 and 8 are Cubes; this Ratio having therefore thoſe cube Numbers for Exponents, the ſimple Ratio, as that of this Progreſſion, which is between each Term, cannot be irrational, † according to Theorem the third.

THEOR. 10. IF *of four Lines in continued Proportion, the Exponents of the Ratio of the Firſt to the Fourth are not Cube numbers,* the Firſt and Second *are Commenſurable only in Power; it is the ſame likewiſe of the Second and Third.* 134

LET ∺ *b. c. d. f.* and that *b. f*::1. 5. the Ratio of *b* to *f* is triple that of *b* to *c*. ‡ Now 1 and 5, the Exponents of the Ratio of *b* to *f*, are not Cubes; therefore the ſimple Ratio of *b* to *c* is Surd: § So likewiſe of the Ratio of *c* to *d*, and that of *d* to *f*. But ſeeing that $b^3. c^3::b. f$; and conſequently $b^3. c^3::1. 5$ ‖ therefore *b* and *c* are commenſurable in power.

THEOREM 11. IF *of four Lines in continued Proportion, the Firſt is not to the Fourth as Number to Number, the Ratio of the Firſt to the Second is not of Number to Number.* 135

∺ *b. c. d. f*; the Ratio of *b* to *f* is triple; therefore this triple Ratio being ſurd, by the ſecond Theorem, the ſimple Ratio of *b* to *c* is ſurd; and ſince that $b^3. c^3::b. f$, this Ratio of *b* to *f* being Surd, that of b^3 to c^3 is Surd. Therefore *c* is not Commenſurable in power with *b*.

PROBLEM 1. TO *find a Line which is incommenſurable in itſelf, and Commenſurable in power with a known or given Line.* 136

LET *b* be a Line, to find another Incommenſurable therewith. The Line *b* being 2, take a Line *x* equal to 3 or to 5, or to any Number which is not a Square. Between theſe two Lines find a mean Proportional, which I call *c*: Hence ∺ *b. c. x.* Now by the ſeventh Theorem the Line *c* is Incommenſurable in itſelf with the two firſt Lines, and commenſurable in power. ¶

PROB. 2. TO *find a Line that is Incommenſurable, as well in itſelf as in power, with a known and given Line.* 137

LET the given and known Line be B, by the preceeding Problem find the Line D, which is in itſelf Incommenſurable thereto. Then, between B and D, having found the mean Proportional Line C; this Line by Theorem the eighth † will be Incommenſurable with B, either in itſelf or in power; *which was to be done.*

¶ *B.* 3. *n.* 83. † *Sup. n.* 124. ‡ *B.* 3. *n.* 83.
§ *Sup. n.* 124. ‖ *B.* 3. *n.* 87.
¶ *Sup. n.* 134. † *Sup. n.* 135.

138 **Theorem 12.** The *Diagonal of a Square is Incommenſurable in itſelf, and Commenſurable in Power with each of the Sides.*

Let the Square be ABCD. It muſt be proved that its Diagonal AC, is incommenſurable in itſelf with AB; but that it is Commenſurable therewith in Power. Let AB$=a$, ſince that BC $=$AB: Then BC$=a$: Therefore $aa+aa=\overline{AC}^2$;* then aa. $\overline{AC}^2$ ::1. 2; thus you ſee the Diagonal is Commenſurable in Power. Now 2 is not a ſquare Number: † Therefore AC, the Root of $\overline{AC}^2$ cannot be expreſſed by Number; ‡ it is therefore Incommenſurable in itſelf; § *which was to be proved.*

139 **Theor. 13.** A *rational Line being divided into mean and extreme Ratio, the two Parts thereof are irrational.* Eucl. 13. Prop. 6.

Let CB be a rational Line divided into mean and extreme Ratio at the Point A; I ſay that the Parts AC and AB are not rational Lines; or, which is the ſame Thing, that they cannot be expreſſed by Numbers. To CB join the Line BD, the half of CB, the Square of the intermediate Part AB joined with BD, makes five times the Square of BD; ‖ therefore theſe two Squares are as 5 to 1. Now 5 is not a ſquare Number: Therefore by the Theorem, ¶ the Line AB+BD is irrational, but BD the half of the rational Line AB, is rational; the intermediate Part AB, muſt therefore be Irrational; † and therefore the leſſer Part AC is Incommenſurable; for if it were commenſurable, AB would be ſo likewiſe. ‡

140 **Corallary.** The *mean Part is Incommenſurable with the whole Line, either in itſelf or in Power.*

Let b, be a Line divided into mean and extreme Ratio, x is the mean Part, and $b-x$ the leaſt Part. ∺ b. x. $b-x$. Now ſince $b-x$ is by the preſent Theorem an irrational Line, then by the eighth Theorem § x, the mean Proportional between b and $b-x$, is incommenſurable with b, either in itſelf or in power,

---

* *Sup. n.* 78. † *Sup. n.* 106. ‡ *Sup. n.* 114. § *Sup. n.* 115. ‖ *Sup. n.* 66.

¶ *Sup. n.* 114. † *Sup. n.* 119. ‡ *Sup. n.* 117. § *Sup. n.* 132.

SECT. VI.

Of the Ratio's between the Chords and Radii of the Circle.

ADVERTISEMENT.

THE Chords of one and the ſame Circle are not one to another as the Arcs whoſe Chords they are: The two Arcs BED and CFD being equal, the Arc BDC is double of either of them. If BC, the Chord of that Arc, was therefore double the Chord BD or of the Chord DC, as the Arc BDC is double the Arc BED, then BC would be equal to BD+CD; which cannot be. * Therefore it cannot be ſuppoſed that the Chords of one and the ſame Circle, are one to another as the Arcs which they are the Chords of. 141

Theor. 1. The *Radius of a Circle is equal to the Chord of ſixty Degrees.* 142

The Angle at the Center of an Hexagon is 60 Degrees, the ſixth Part of 360 Degrees, which make the four right Angles that may be conceived about the Center. Then let BAC be an Angle of 60 Degrees; hence BC is the Side of the Hexagon, which muſt be proved equal to the Radius AB or AC. The Triangle BAC is Iſoceles; therefore the Angles at the Baſe are equal; but that of the Vertex is 60, therefore both together make 120, each therefore 60. The Triangle is then Equilateral; therefore BC=AB, or BC=AC; *which was to be proved.*

Corallary 1. *A Circle being given to deſcribe an Hexagon.* Eucl. 4. Prop. 15. 143

The Compaſſes being opened to the Extent of the Radius, it will divide the Circle into ſix equal Parts.

Corallary 2. It *is eaſy to inſcribe an equilateral Triangle in a Circle.* 144

For two Parts of the Hexagon's ſix, make the third Part of the Circle.

Corallary 3. The *Ratio of the Circle's Circumference to its Radius, is above* 3 *to* 1. 145

Each Side of the Hexagon being equal to the Radius, the ſix Sides are equal to three times the diameter; hence the Circumference of the Polygon is to the Radius of its circumſcribing Circle, as 3 to 1. Now the Circumference of the Circle is greater than that of the Hexagon, each Portion of the Circle being greater than the Side of the Hexagon, which is the Chord thereof.

Theorem

---

* *B.* 1. *n.* 12.

146 THEOREM 2. THE *Square of one of the Sides of an equilateral Triangle inſcribed in a Circle, is triple the Square of the Semi-diameter or Radius of the Circle.* Eucl. 13. Prop. 12.

LET ABC be an equilateral Triangle inſcribed in the Circle X; let there be drawn the Diameter AD which cuts the Chord BC perpendicularly into two equal parts. * It muſt be proved that the Square of AB is triple that of the Radius; ſince that BC is the Chord of the third Part of the Circle, BD is the Chord of the ſixth Part, and therefore equal to the Radius, of which the Diameter AD is double. † hence if BD$=b$, then AD$=2b$, and $\overline{AD}^2=4bb$. Let AB$=a$; ſince that ABD is a right Angle: then $aa+bb=\overline{AD}^2$. ‡ Therefore $aa+bb=4bb$. Taking away bb from each Side, there will remain $aa=3bb$; Q. E. D.

ADVERTISEMENT.

147 THERE are ſeveral ſuch like Theorems, which I ſhall not give Place to in theſe Elements. It is evident that the Tangent of an Angle of 45 Degrees is equal to Radius: For it makes with the Radius a rectangled Triangle, whoſe Baſe is the Secant, the Angles at the Baſe are equal, (*viz.*) each of 45 Degrees: Therefore, the Tangent and the Radius are equal. 2. The Square of the Secant is equal to the Square of Radius and of the Tangent, ǂ or which is the ſame, to twice the ſquare of Radius. Now the Chord of ninety is equal to the Secant of forty-five Degrees: Conſequently the Chord of ninety is equal to twice the Square of Radius. as is evident.

148 LEMMA 1. IF *the Angles at the Baſe of an Iſoceles Triangle, are each double that of the Vertex, I ſay that the Line which divides one of the Angles at the Baſe into two equal Parts, will divide the oppoſite Side into mean and extream Ratio.*

LET the Iſoceles Triangle be BAC, and CD the Line which divides BCA, one of the Angles at the Baſe into two equal Parts; it muſt be proved that it divides AB into mean and extream Ratio at the Point D. 1. Seeing that the Angle BCA is ſuppoſed to be double of BAC; then the half DCA is equal to the Angle CAD: Wherefore the Triangle ADC having the Angles at the Baſe AC equal, it is Iſoceles, § and the Sides AD and DC will be equal. 2. The Angle BDC is equal to the two oppoſite ones DAC and ACD; * conſequently it is equal to the Angle ACB which make theſe two Angles, and to DBC which is by the Hypotheſis equal to ACB; therefore the Triangle DCB having its Angles at the Baſe DB equal, is alſo Iſoceles; Hence DC$=$BC. 3. The two Iſoceles Triangles BAC and BDC, which have at the Point B an Angle common, † are equiangular, and therefore ſimilar: Then ‡ AB is to BC, or to AD its equal, as DC or its equal AD is to DB, *that is,* ∺ AB. AD. DB; and conſequently AB is divided into mean and extream Ratio; ſince that the Part AD is a Mean between the whole AB, and the other Part DB. ⸸

---

* *B.* 1. *n.* 90. † *Sup. n.* 142. ‡ *Sup. n.* 78. ǂ *Sup. n.* 78.
§ *B.* 2. *n.* 83. * *B.* 2. *n.* 73. † *B.* 2. *n.* 85. ǂ *Sup. n.* 10.

Lemma 2. To *deſcribe an Iſoceles Triangle, having each of its Angles at the Baſe double the other.* Eucl. 4. Prop. 10. 149

Having the Line AB, divide it into mean and extream Ratio at the Point D; § from B and D, with the Extent of the mean Part AD, deſcribe two Arcs which interſect at the Point C. Hence BC=DC=AD. The two Triangles ADC and DCB are therefore Iſoceles. By the ſame Conſtruction ∺ AB. AD. DB. ſubſtituted in the Room of equal Lines AB. BC :: BC. BD; then BAC and DBC are ſimilar, or equiangular * and Iſoceles. Now the Angle BDC is equal to DAC+ACD the oppoſite interior Angles which are equal, ſeeing ADC is Iſoceles: Therefore DBC equal to BDC is the Double of BAC; *which was to be proved.*

Lemma 3. In *an Iſoceles Triangle, whoſe Vertex is at the Centre of the Circle, and its Baſe the Side of a Decagon, each Angle at the Baſe is double that at its Vertex.* 150

The Iſoceles Triangle BAC hath its Vertex at the Centre of X, its Baſe BC is the Side of a Decagon, and conſequently the Chord of an Arc of 36 Degrees the tenth Part of 360 Degrees which make the whole Circle. It muſt be proved that each Angle at the Baſe CBA and ACB, contains 72 degrees, the Double of 36; which is evident, for the three Angles make 180 Degrees, † if from this Number be taken 36, the Content of the vertical Angle, 144 remains for the Angles at the Baſe, which being equal, each is 72 the Double of 36.

Theorem 3. The *Radius being divided into mean and extream Ratio, the mean Part is the Side of a Decagon, or Chord of thirty ſix Degrees.* Eucl. 4. Prop. 10. (*Fig. above*) 151

Let AC be the Radius of a Circle divided into mean and extream Ratio at the Point D, ‡ and that the Chord BC is equal to the mean Part AD, then the Angles at the Baſe BC are each double of BAC. § Therefore by the preceeding Lemma, BC is the Chord of the twelfth Part of the Circle, and conſequently the Side of the Decagon.

Lemma 4. Two *Lines being drawn from one of the Angles of a Pentagon to the Ends of the oppoſite Side, they make an Iſoceles Triangle, and each of the Angles at the Baſe is double the Angle at the Vertex.* 152

Z is a regular Pentagon, having drawn from A two Lines to the Extremities of the oppoſite Side BC, they make the Triangle BAC, which muſt be proved to be Iſoceles, and that ABC and BCA are each double of BAC. 1. AB and AC the Chords of

equal

⸸ *Sup. n.* 34. § *Sup. n.* 34. * *Sup. n.* 13.

† *B.* 2. *n.* 75. ‡ *Sup. n.* 34. § *Sup. n.* 149.

equal Arcs, are equal: Therefore BAC is Iſoceles. * 2. The Angle BAC is meaſured by half the Arc BC, and the Angle ACB by half the Arc AEB. § Now AEB is double of BC: Therefore the Angle ACB is double the Angle BAC. It is in like manner proved that ABC, is double of BAC.

153 THEOREM 4. IF *in a Pentagon two lines interſect, each of which is the Chord of double the Arc ſubtended by each Side of the Pentagon, they are divided into mean and extream Ratio, and the mean or longeſt Part is one of the Sides of the Pentagon.*

SUPPOSE Z to be a regular Pentagon inſcribed in a Circle, the Lines BD and EF interſect, each being the Chord of double the Arc ſubtended by each of the Sides of the Pentagon, it muſt be proved that they are cut into mean and extream Ratio, and that the mean or longeſt Part, ſuch as BC, is one of the Sides of the Pentagon. 1. The Angle BEF is meaſured by half the two Arcs BG and GF, † and the Angle BCE, by half the two Arcs BE and FD. ‡ Now theſe two Meaſures are equal: therefore the two Angles are equal; hence EBC is Iſoceles, ‖ and therefore BE=BC: Therefore BC is one of the Sides of the Pentagon; *which was to be proved.* 2. By the ſame Reaſon CF=FD: And ſince BD=FE are Chords of the ſame Angles, it muſt be that CE=CD; Therefore ECD is Iſoceles. The Triangle BED is alſo Iſoceles, ſeeing that BE=ED; for Z is ſuppoſed to be a regular Pentagon: The two Iſoceles have one Angle common at D, therefore they are equiangular: * Wherefore as BD is to BE, or its equal BC; ſo ED or its Equal BC is to CD, § *that is,* ÷ BD. BC. CD; and conſequently, according to the Definition of mean and extream Ratio, BD is divided as was propoſed.

154 THEOR. 5. THE *right Line compounded of the Side of an Hexagon and that of a Decagon both inſcribed in one and the ſame Circle, is divided into mean and extreme Ratio.* Eucl. 13. Prop. 9.

LET the Side of the Decagon be AB, and the Side of the Hexagon BE, *that is,* a Line equal to the Radius BD; * I ſay that AE is divided into mean and extreme Ratio at the Point B. EBD is Iſoceles, ſeeing that BE is ſuppoſed equal to the Radius BD; the Triangle ABD is alſo Iſoceles, and the Angle BAD or ABD is the double of BDA. ‡ Now DBA is alſo the double of BDE, as being equal to the two equal Angles BED and EDB: ‖ therefore BED, and BDA are equal one to another, and taken together they are equal to EAD; hence the Triangle AED is Iſoceles; wherefore, ſeeing that BD divides the Angle EDA into two equal Parts, EA is therefore divided into mean and extreme Ratio. * *Q. E. D.*

---

* *B.* 1. *n.* 31. & *B.* 2. *n.* 57. § *B.* 2. *n.* 39. † *B.* 2. *n.* 39.
‡ *B.* 2. *n.* 52. ‖ *B.* 2. *n.* 83.
* *B.* 1. *n.* 85. § *Sup. n.* 10. * *Sup. n.* 142.
‡ *Sup. n,* 150. ‖ *B.* 2. *n.* 73. * *Sup, n.* 148.

CORALLARY. FROM *hence it follows that if AE is divided into mean and extreme Ratio, and that its Segment AB is the Side of a Decagon, BE will be the Side of an Hexagon inſcribed in the ſame Circle.* 155

SEEING that there can be but one mean Proportional between AE and AB, and it has been proved that BE, the Side of the Hexagon is that mean.

PROBLEM 1. A *Circle being given, to find the Chord of* 36 *Degrees, or Side of a Decagon.* See *fig.* To Num. 150. 156

THE Circle is X, and the Radius is AB, which divide into mean and extreme Ratio at D. * Take the Chord BC equal to AD, which is the Side of the Decagon required. †

PROB. 2. To *deſcribe a Decagon upon a Line given for a Side.* 157

DIVIDE the given Side BC into mean and extreme Ratio, ⸸ and join thereto the mean Part; which gives a new Line AC, divided likewiſe into mean and extreme Ratio, whoſe mean part is BC; ‡ deſcribe a Circle having AC for its Radius, then will BC be the Side of its inſcribed Decagon; § whereby the whole may be eaſily conſtructed.

PROB. 3. A *Circle being given, to find the Side of a Pentagon, or to Inſcribe a Pentagon in a Circle.* Eucl. 4. Prop. 11. 158

1. HAVING found the Side of the Decagon, ‖ that of the Pentagon is the Chord of double the Arc, ſubtended by one of the Sides of the Decagon. 2. The Side of the Pentagon may alſo be found thus. Deſcribe an Iſoceles Triangle, having its Angles at the Baſe each double that at the Vertex, * and either inſcribe that *(or one Similar thereto)* in the given Circle, † which will be the Side of the Pentagon ſought. ‡

PROB. 4. To *deſcribe a Pentagon upon a given Side.* 159

LET the given Side be ED, ſuppoſe the Pentagon to be already made, then will the Method of finding the Lines BD and EF, diſcover what muſt be done. Now ſeeing that BD is divided into mean and extreme Ratio at the Point C, § that BC equal to BE, and conſequently to ED, is the mean part, by dividing the Side ED into mean and extreme Ratio, and adding thereto the mean Part, there will ariſe a new Line equal to BD, ‖ divided likewiſe into mean and extreme Ratio, whoſe mean Part is ED, conſequently equal to BC. Then as the three Sides of the Triangle DEB are known, it is eaſy to compleat the whole Pentagon, by circumſcribing a Circle about the Triangle. *

PROB. 5. To *Circumſcribe a regular Pentagon about a Circle.* Eucl. 4. Prop. 12. 160

---

* *Sup. n.* 34. † *Sup. n.* 151. ⸸ *Sup. n.* 34. ‡ *Sup. n.* 35. § *Sup. n.* 156. ‖ *Sup. n.* 156. * *Sup. n.* 149. † *B.* 2. *n.* 100. ‡ *Sup. n.* 152. § *Sup. n.* 153. ‖ *Sup. n.* 35. * *B.* 1. *n.* 87.

1. In the given Circle inſcribe a Pentagon. 2. Draw right Lines from the Center of the Circle thro' the five Angles of the Pentagon. 3. Draw Tangents thro' the Points of the Circumference where theſe Lines croſs the Circle. Theſe Tangents, *as is evident*, will form the Pentagon required.

161 Prob. 6. To *inſcribe a Circle, in a regular Pentagon.* Eucl. 4. Prop. 13.

Upon two of the Sides of the Pentagon, raiſe Perpendiculars, the Point where they interſect is the Center of the Circle required, having for its Radius the Extent between the Center and the Middle of one of the Sides of the Pentagon, which is evident.

162 Prob. 7. To *circumſcribe a Circle about a Pentagon.* Eucl. 4. Prop. 14.

Having by the preceeding Problem found the Center of the Circle, take the diſtance between the Center and one of the Angles of the Pentagon, a Circle deſcribed with that extent will be the Circle required.

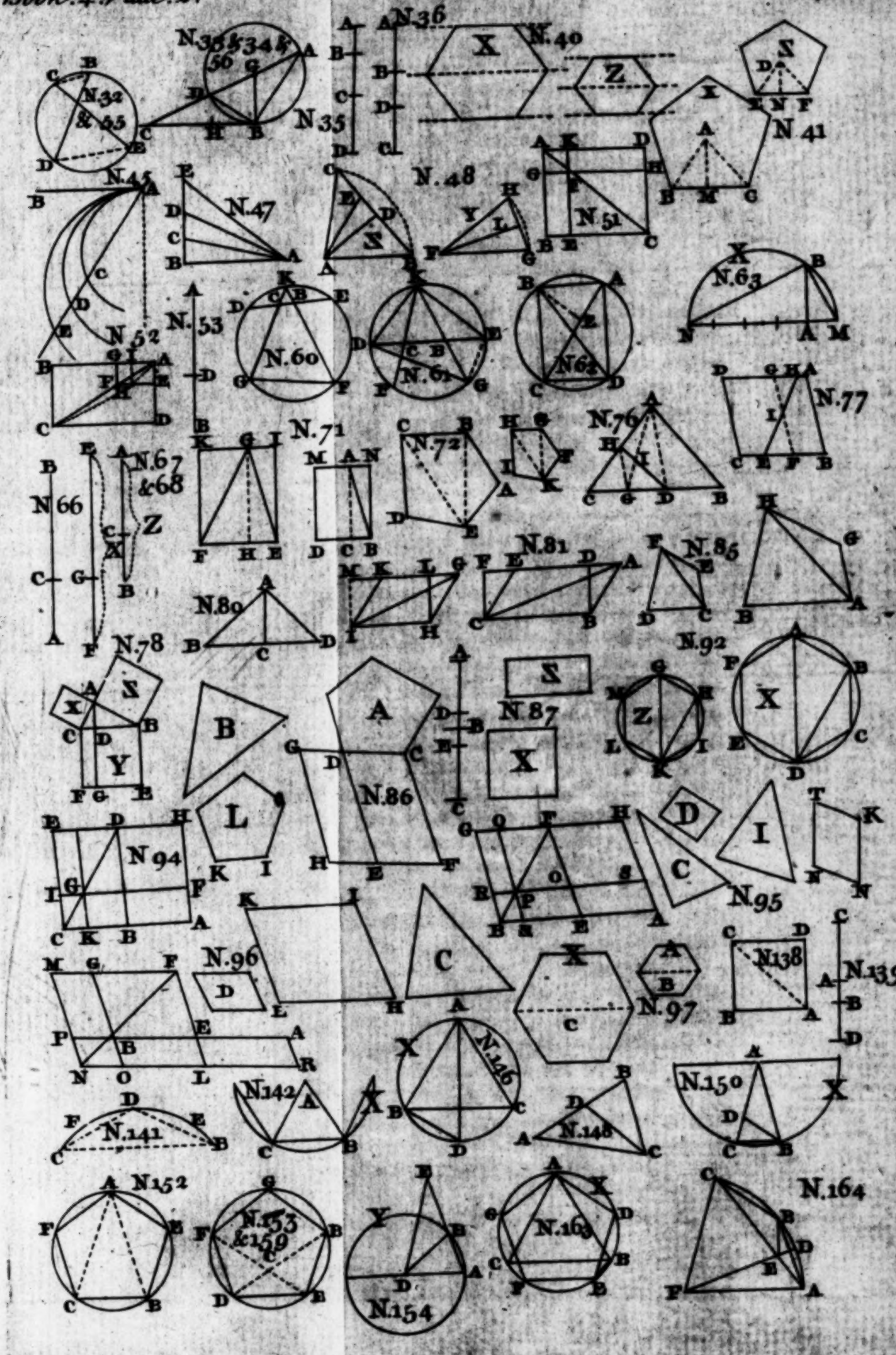
163 Prob. 8. To *inſcribe in a Circle a Quindecagon, or Polygon of Fifteen Sides.* Eucl. 4. Prop. 16.

The Circle X being given to inſcribe therein a Quindecagon, 1. Divide it into ſix equal Arcs; the Chord of each of thoſe Arcs is the Radius thereof. 2. Take two of theſe Arcs, the double whereof is the third Part; the Chord thereof is therefore a Side of the Equilateral Triangle ABC. 3. Make the Pentagon ADEFG, each Side of which is the Chord of the Arc of the Circle's fifth Part; the Chord of the third Part of this Arc, is the Side of the Quindecagon, five Sides of which make the third Part of the Circle, and conſequently anſwer to the Side of the Equilateral Triangle. This being ſo, BE is the Side of the Quindecagon required; for three parts of the Polygon anſwer to AD, and as many to DE. Hence ſix Sides will anſwer to AD and DE: But five Parts anſwer to AB; therefore BE, the ſixth Part of AE, remains for the Side of the Quindecagon.

164 Theorem 6. The *Square of one of the Sides of a Pentagon is equal to the Squares of one of the Sides of a Decagon, and of a Hexagon, inſcribed in the ſame Circle.* Eucl. 13. Prop. 10.

Let AC be the Side of a Pentagon. Having divided the Arc AC into two equal parts at the Point B, and drawn the Chords AB. BC, they are the Sides of the Decagon, and AF or FC Radii of the Circle, are the Sides of the Hexagon. It muſt be proved that $\overline{AC}^2 = \overline{AF}^2 + \overline{AB}^2$. Let there be drawn the Line FD dividing AB into two equal Parts, and from the Point E let there be drawn EB; there will be the two Triangles ACF and ECF which are Similar; for they have the Angle C common; and the Angle CFE is equal to FAC; becauſe they both contain an equal number of Degrees, *viz.* 54, for the Arc BC being of 36

Degrees


N.32 & 55
N.33 & 34 & 56
N.35
N.36
N.40
N.41
X
Z
N.45
N.47
N.48
N.51
N.52
N.53
N.60
N.61
N.62
N.63
N 66
N.67 & 68
N.71
N.72
N.76
N.77
N.78
N.80
N.81
N.85
N.86
N.87
N.92
N 94
N.95
N.96
N.97
N.138
N.139
N.141
N.142
N.146
N.148
N.150
N.152
N.153 & 159
N.154
N.163
N.164

Degrees and BD of 18, DC will be of 54, the Meaſure of the Angle CFD, to which CAF is equal becauſe the Triangle AFC is Iſoceles, and the Angle of the Center is 72 Degrees. Then theſe Triangles being Similar ∺ AC. AF. EC: * Therefore $\overline{AF}^2=AC\times EC$. † Now the Iſoceles Triangle AEB is Similar to ABC which is alſo Iſoceles; for they have one Angle common, *viz.* A, and conſequently all the others equal. ‡ Then ∺ AE. AB. AC, then $\overline{AB}^2=AE\times AC$; then $\overline{AF}^2+\overline{AB}^2=AC\times EC+AE\times AC$. But theſe two Rectangles are equal to the Square of AC; § for AE+EC=AC. Therefore $\overline{AC}^2=\overline{AF}^2+\overline{AB}^2$; *Q. E. D.*

CORALLARY. THE *Square of the Side of a Pentagon inſcribed in a Circle, and the Square of the Chord which ſubtends the Angle of the Polygon joined together, are equal to five times the Square of the Radius of the Circle.* 165

LET BF the Side of the Pentagon be equal to *b*, and BC the Diameter of the Circle $=2a$, and BD the Subtenſe of the Angle F of the Pentagon $=d$. It muſt be proved that $\overline{BF}^2+\overline{BD}^2=5\overline{AB}^2$, or $bb+dd=5aa$. Let there be drawn DC $=c$, which will be the Side of the Decagon, and will form the right angled Triangle BDC. ‖ Then $dd+cc=4aa$; * but by the preceeding Theorem $bb=aa+cc$; this Equation added to the Firſt, the Sum will be $bb+dd+cc=aa+cc+4aa$; ſubtract *cc* from each Side, $bb+dd=5aa$; *which was to be proved.*

PROB. 9. *A Circle being given, to find the Side of a Pentagon and of a Decagon, by a Method different from the Former.* 166

LET the Diameter of the given Circle be AB, and its Radius CD Perpendicular upon AB; if there be taken CE half of BC, and that from the Point E be drawn ED, and that afterwards be taken EF=ED and there be drawn DF; I ſay that this Line will be the Side of the Pentagon required. For $BF\times FC+\overline{EC}^2=\overline{EF}^2=\overline{ED}^2$. † But $\overline{FD}^2=\overline{DC}^2+\overline{FC}^2$, ‡ hence $BF\times FC+\overline{EC}^2=\overline{DC}^2+\overline{EC}^2$; then taking away $\overline{EC}^2$ from each Side, $BF\times FC=\overline{DC}^2=\overline{BC}^2$; and this Equation reduced to a Proportion, § it will be ∺ BF. BC. FC; therefore the Line FB is divided into mean and extreme Ratio; ‖ and ſeeing BC or DC is the Side of the Hexagon, FC will be that of the Decagon. * But $\overline{FD}^2=\overline{DC}^2+\overline{CF}^2$; † therefore

---

* *Sup. n.* 10. † *B.* 3. *n.* 57. ‡ *B.* 2. *n.* 85.
§ *B.* 3. *n.* 18. ‖ *B.* 2. *n.* 44.
* *Sup. n.* 78. † *B.* 2. *n.* 22. ‡ *Sup. n.* 78.
§ *B.* 3. *n.* 58. ‖ *Sup. n.* 34.
* *Sup. n.* 154. † *Sup. n.* 78.

therefore DF is the Side of the Pentagon; * *which was to be found.*

OR THUS

167 Take BC equal to the Radius of the Circle, BD equal to the Chord of Ninety, or to the Side of a Square inſcribed in the Circle, and CE equal to BD. The Line DE will be the Side of the Pentagon, and AD being the Radius of the Circle, AE will be the Side of the Decagon. To demonſtrate the Truth thereof, let fall from C a Perpendicular upon the Radius AB, and from D upon the Center A, the Perpendicular AD; hence BD is the Chord of Ninety, to which CE is equal. Suppoſe AB, AD or $BC=2a$; hence $BG=4a$. Now ∻ BG. BC. BF. † *that is,* ∻ $4a.\ 2a.\ a$; then BF is half the Radius. But it appears by the preſent Theorem, that to find ED the Side of the Pentagon, there is made $FE=FD$, if then it be proved that there is this Equality in the preſent Conſtruction, it is a ſufficient Demonſtration. Since $FB=a$, FG will be $=3a$, the Rectangle $GF\times FB$, *that is,* $3aa=\overline{FC}^2$. ‡ Now AB and AD being each $2a$, their Squares together are $8aa=\overline{DB}^2=\overline{CE}^2$ by Conſtruction, § equal to $\overline{CF}^2+\overline{EF}^2$. If then from $\overline{CE}^2=8aa$ be taken $\overline{FC}^2=3aa$, there will remain $5aa$ for $\overline{EF}^2$, to which $\overline{FD}^2$ is alſo equal ſeeing it is equal to $\overline{DA}^2+\overline{AF}^2$, ‖ *that is,* to $4aa+aa$: Wherefore $FD=FE$; *Q. E. D.*

168 THEOR. 7. IF *three Angles of an Equilateral Pentagon being taken any how are Equal, it is Equiangular.* Eucl. 13. Prop. 7.

LET the Equilateral Pentagon be ABCDE, which hath three of its Angles equal, take them how you will, *viz.* A. C. D; it muſt be proved that they are all equal, that $C=B=E$. The Triangles BAE, BCD CDE being Iſoceles, whoſe equal Angles A, C, D are comprehended by equal Sides, they are every where equal; * and therefore the Baſes BE, BD, CE are equal, as alſo all the Angles formed by equal Sides upon the ſaid Baſes are equal. So likewiſe the two Iſoceles Triangles BEC, EBD having all their Sides equal, they have alſo the Angles EBC, ECB, BED, BDE equal. † If to each of the equal Angles EBC, ECB the equal ones ABE, DCE be added. the Total B will be equal to the Total C. The ſame for the Angles E and D, therefore the five Angles are all equal; *which was to be proved.*

---

* *Sup. n.* 164. † *Sup. n.* 28. ‡ *Sup. n.* 28 & *B.* 3. *n.* 57.
§ *Sup. n.* 78. ‖ *Sup. n.* 78.

* *B.* 2. *n.* 98. † *B.* 2. *n.* 92.

THEOREM 8. WHEN *the Radius of a Circle is Rational, the Side of a Decagon inſcribed in the Circle is incommenſurable with the Radius, both in itſelf and in power.* 169

LET B the Radius of a Circle be a rational Line, divided into mean and extreme Ratio, x which I ſuppoſe to be the mean Part, will be the Side of the Decagon. * Now the mean Part x is incommenſurable with B, both in itſelf and in power. †

THEOREM 9. WHEN *the Radius of a Circle is rational, the Sides of the Pentagon inſcribed in the Circle are Incommenſurable with the Radius, either in themſelves or in power.* Eucl. 13. Prop. 11. 170

LET the rational Line b, be the Radius of a Circle, z the Side of a Pentagon, and x the Side of a Decagon, both inſcribed in the Circle whoſe Radius is b. Therefore $bb+xx=zz$: ‡ then ſeeing that the Square xx is Irrational, as we demonſtrated in the laſt Theorem, the Square zz cannot be rational, § conſequently its root z is Irrational, ſince that every Square which is not equal to a Square Number, cannot have for its Root a Quantity exactly equal to any Number, as we have proved above. ‖

OF *the Quadrature of the Circle.*

WE have ſhewn * that the Surface of the Circle is equal to a Triangle, having for its height the Radius of the Circle, and the Circumference thereof for its Baſe; therefore a mean Proportional between the Radius and the Circumference, is the Side of a Square equal to the Circle. But to find this mean Proportional, the Radius being given, a right Line muſt be found equal to the Circumference, which cannot be done without knowing the Ratio of the Radius to the Circumference, which is known nearly, but not exactly. The Circle may be taken for a Polygon whoſe Sides are infinite; therefore the moſt exact and leaſt deficient Method of finding the ſaid Ratio, is to take the greateſt Polygon poſſible; and to ſee what is the Ratio of its Circumference with the Diameter of its circumſcribing Circle. *Archimedes* conſidered a Polygon of Ninety ſix Sides, the Circumference of which to the Diameter of the Circle is as 223 to 71, which Ratio is leſs than that of the Circumference of the Circle to the Diameter, the inſcribed Polygon being leſs than the Circle. † Comparing the ſame Polygon, but conſidered as a Circumſcribing one with the Diameter, he found that the Ratio of one to the other was as 22 to 7, which is greater than that of the Circle's Circumference to its Diameter; by reaſon a circumſcribing Polygon is greater than the Circle. Here 171

---

* *Sup. n.* 151. † *Sup. n.* 140. ‡ *Sup. n.* 164. § *Sup. n.* 116. ‖ *Sup. n.* 114.

* B. 2. *n.* 154. † *B.* 3. *n.* 147.

Here is therefore two Ratio's (*viz.*) that of 223 to 71, and that of 22 to 7, one of which is leſſer, and the other greater than the true one required. Reduce theſe two Ratio's to one and the ſame Conſequent. *

1561 } 1562 } 497. Having therefore ſuppoſed the Diameter of 497 Parts, the Circle's Circumference will be greater than 1561, and leſs than 1562. Dividing unity which is the difference of theſe two numbers by 497, the Quotient $\frac{1}{497}$ ſhews that the Difference which each number differs from the true Length of the Circumference, is leſs than $\frac{1}{497}$ part of the Diameter; which is but a ſmall matter.

172 Thoſe who have taken Polygons containing more than 96 Sides, have found an exacter Ratio. *Pell* conſidered one of 256 Sides, each of which is leſs than 24545, ſuppoſing the Radius of the Circle to be of 1000000; therefore the Semi-circumference of the Polygon is leſs than 3141760. If the Diameter of the Circle was therefore 1000000, the whole Circumference of the circumſcribed Polygon would be to the Circle leſs than 314176. Now the Calculus contained in a Work printed at Amſterdam in the Year 1647, ſuppoſes no extraction of the Root; and is grounded upon the following Theorem.

Theorem.

BC being the Tangent of an Arc leſs than forty five Degrees, and BD the Tangent of a double Arc; BC is to BD as the Square of the Radius AB, leſs the Square of BC, is to twice the Square of the Radius AB. It muſt be demonſtrated that $BC.\ BD :: \overline{AB}^2 - \overline{BC}^2.\ 2\,\overline{AB}^2$. Seeing the Angle BAD is ſuppoſed double BAC; then $AB.\ AD :: BC.\ CD$, † and *compoundedly*, $AB.\ AD+AB$ (or DE) $:: BC.\ BD$. ‡ Wherefore $\overline{AB}^2.\ \overline{DE}^2 :: \overline{BC}^2.\ \overline{BD}^2$. § *Alternately*, $\overline{AB}^2.\ \overline{BC}^2 :: \overline{DE}^2.\ \overline{BD}^2$; then *dividedly, and compoundedly*, $\overline{AB}^2 - \overline{BC}^2.\ 2\overline{AB}^2 :: \overline{DE}^2 - \overline{BD}^2.\ 2\overline{DE}^2$. Hence it remains to be demonſtrated that $\overline{DE}^2 - \overline{BD}^2.\ 2\overline{DE}^2 :: BC.\ BD$, for then $\overline{AB}^2 + \overline{BC}^2.\ 2\overline{AB}^2 :: BC.\ BD$. Seeing that two Ratio's equal to a Third are equal one to another. Now $\overline{DE}^2 = \overline{AE}^2 + 2AE \times AD + \overline{AD}^2$. ‖ But $\overline{AD}^2 = \overline{AB}^2$ (or $\overline{AE}^2 + \overline{BD}^2$.*)then $\overline{DE}^2 = 2\overline{AE}^2 + 2AE \times AD + \overline{BD}^2$.

---

* *B.* 3. *n.* 89. † *Sup. n.* 17. ‡ *B.* 3. *n.* 49.
§ *B.* 3. *n.* 84. ‖ *B.* 3. *n.* 20.
* *B.* 3. *n.* 140.

Then ſubtracting $\overline{BD}^2$ from each Side $\overline{DE}^2-\overline{BD}^2=2\overline{AE}^2+2AE\times AD$. Now $2\overline{AE}^2+2AE\times AD=2AE\times ED$: * Then $\overline{DE}^2-\overline{BD}^2=2AE\times ED$. Theſe two Rectangles $2AE\times ED$ and $2\overline{ED}^2$ have the ſame Baſe, *viz.* ED; they are therefore one to another as AE or AB is to ED, Now AB. ED::BC. BD; therefore $\overline{DE}^2-\overline{BD}^2$ equal to $2AE\times ED$ is to $2\overline{ED}^2$ as BC to BD; *which was to be ſhewn.*

173 According to the Archimedian Proportion of the Circumference of the Circle to its Diameter, *viz.* That of 22 to 7, or 44 to 14, or 66 to 21, which are each the ſame, the Surface of the Circle is to the Square of its Diameter as 11 to 14. For let the Circumference of the Circle be called m, and the Diameter n, multiply m and n by n, then mn. nn. m. n. † and ſeeing that m. n::44. 14; then mn. nn; 44. 14, therefore the fourth Part of mn is to nn as 11, the fourth Part of 44, is to 14. Now the Quarter of mn is the Surface of the Circle. ‡ this Surface is therefore to nn, the Square of the Diameter, as 11 to 14.

174 Hence the Quadrature of the Circle would have been found, if the true Ratio of the Circumference to the Diameter had been, as 22 to 7; but this Ratio is not exact, and what is the true one is not yet known. Nevertheleſs this Quadrature is partly known, *that is*, a Portion of a Circle may be aſſigned equal to a rectilinear Figure; for the Triangle ABC being right Angled, the Semi-circle ADBE is equal to the two Semi-circles AGB and BFC. § then taking away the Parts which are common, *viz.* ABD and BCE, there will remain AGBDA and CEBF, which are equal to the Remainder of the Semi-circle ADBEC, which remainder is the Triangle ABC. It alſo appears to the Eye that the Figure AEBFCDGH, is equal to the Square ABCD; for by conſtruction the Parts AHD=AEB=BFC=CGD.

175 The Figures themſelves ſhew that a Triangle circumſcribed about a Circle is Quadruple of the inſcribed one, that a circumſcribed Square is double the inſcribed one, that the circumſcribing Hexagon is to the inſcribed one as 4 to 3. Obſerve here by the Way, that to divide a Line as BE Geometrically into three equal Parts, make the Side of an equilateral Triangle inſcribed in a Circle, then having made the Hexagon and divided the Sides thereof by Perpendiculars, you will find that the Perpendiculars will divide BE into three equal parts BC, CD, DE.

---

* *B.* 3. *n.* 29. † *B.* 3. *n.* 54. ‡ *B.* 2. *n.* 154.
§ *Sup.* *n.* 79.

The following is a general Theorem which affords a very extenſive Idea, of what may be known in this matter.

176 WHEN *of two inſcribed Polygons one has twice as many Sides as the other, the Greater is to the Leſſer, as the Radius of the Circle is to the Apotome of the leſſer Polygon.*

Z is a Polygon whoſe Apotheme is AB, and X a Polygon having twice the Number of Sides. The Surface of Z is equal to as many times the Triangle DAE, as Z hath Sides; and it is plain, that the Triangle DCE being added to each of theſe Triangles, will give the Surface of X, which is to that of Z, as the Rhombus AECD is to the Triangle DAE. Now theſe two Figures are one to another as the Radius AC is to AB, which is the Apotome of Z; for the Surfaces of the two Triangles DAE and DCE, which have the ſame Baſe, are one to another as their Heights AB and BC; ſo of all other Polygons, of which one has twice as many Sides as the other. If the Polygon X is divided ſo that it may have a double Number of Sides, which I call Y; it will be the ſame: And it is proper to obſerve, that the firſt Polygon Z, is to the Third Y, as the Rectangle of the Apotomes of the Firſt and Second is to the Square of the Diameter; which I demonſtrate thus. Let the Apotome of the Firſt be called m; that of the Second n; and the Radius k, according to what has been demonſtrated.

$$\begin{cases} Z.\ X :: m.\ k. \\ X.\ Y :: n.\ k. \end{cases}$$

The Antecedents being multiplied by the Antecedents, and the Conſequents by the Conſequents, $ZX.\ XY :: mn.\ kk.$ Now $ZX.\ XY :: Z.\ Y$; then $Z.\ Y :: mn.\ kk.$ The firſt Polygon Z is to the Third Y, as mn, the Rectangle of the Apotomes of the Firſt and Second is to kk the Square of Radius.

177 The ſame Way it is demonſtrated that the firſt Polygon is to the Fourth, as the Solid produced by the Apotomes of the Firſt, Second and Third is to the Cube of the Radius of the Circle, and ſo of the Reſt; *ad. inf.* The Apotome of an inſcribed Square is the half of one of its Sides; hence a Square is to an Octagon as the half of one of its Sides is to the Radius of its circumſcribing Circle; or which is the ſame, as one of the Sides is to the Diameter of the Circle. Wherefore it is juſtly concluded, that an inſcribed Square is to a Polygon of infinite Sides, doubling always the Sides, *that is*, of a Square is made an Octagon, of an Octagon a Polygon of ſixteen Sides, and ſo on which Polygon may be conſidered as a Circle; it may, I ſay, be concluded that the Square is to the Circle as a Quantity

produced by the Multiplication of all the Apotomes of the Polygons, to the higheſt Power of the Radius of the Circle. But this is knowing ſcarce any thing; for beſides the ſurd Ratio's which we frequently meet with, the Radius and Diameter are diviſible *ad. inf.* Therefore the Number of all the Apotomes cannot be comprehended. In fine it is therefore impoſſible by this way to reach from a Polygon to a Circle; this Figure is therefore incomprehenſible, altho' it be the Simpleſt of all Geometrical Figures.

BOOK. V.

Of Solids, or the third Species of Extenſion.

SECT. I.

Of the Sections and meeting of Planes.

ADVERTISEMENT.

Every Solid of what Sort ſoever it be, may be conceived as compoſed of different Planes placed one upon another; or as deſcribed by the Motion of one only, moving in a certain manner. Alſo according as two or more Planes interſect or meet, they form Solids; which obliges us in treating hereof, to begin with the meeting and Section of Planes.

N. B. *The References to the paſte board Plates annexed to this Work are denoted by the Letters* P. B.

Evident Propoſitions concerning Planes.

1 Proposition 1. A *right Line in moving along another right Line, which is at reſt, and keeping with it one and the ſame Poſition, deſcribes a ſuperficial Plane, or ſimply a Plane.*

This Superficies hath all its parts equally placed between its Extremities; for they are uniformly deſcribed: And that which makes the Difference of other Superficies, which are not Planes, is the Way by which they are generated, a right Line may be every way applied to it, as is going to be declared.

2 Prop. 2. A *right Line may be every way applied to a plane Surface, and will agree with it.*

3 Prop. 3. The *Surface of a Plane is the Shorteſt that can be conceiv'd between the Ends or Bounds thereof.*

for a Plane is compoſed of right Lines, which are the Shorteſt that can be conceived between their Extremities.

4 Prop. 4. A *Plane raiſed upon a Plane is Perpendicular thereupon, when it inclines no more to one Side thereof than to the other.*

5 Prop. 5. Two *Planes are Parallel, when all their Parts are at an equal Diſtance one from the other, and which being produced will never meet.*

Prop. 6. *A Plane may be produced, or conceived as produced* 6
from all its Sides, as much as ſhall be requiſite.

Prop. 7. A *right Line cannot be part upon a Plane, and part* 7
above it. Eucl. 11. Prop. 1.

For if it were, then this Line cannot be applied to the Plane and agree with it, therefore according to the ſecond Propoſition, the Plane would not be a Plane. Beſides this, it may be conceived that the Part of this Line which is on the Plane being prolonged, will always remain on the Plane, is therefore a different Line from that which is out of it; notwithſtanding theſe two Lines having two Points common, they cannot be different. *

Prop. 8. Two *right Lines that interſect, may be conceived in* 8
one and the ſame Plane. Eucl. 11. Prop. 2.

The Lines DB and EC interſect; having drawn right Lines between AB and AC, there will be a Superficies, which is a Plane: for a right Line may be applied to it, and this Superficies is the Shorteſt that can be conceived between the Lines AB and AC which are upon the Plane; which being produced, if AD and AE which are parts of the Lines BD and EC, be not found in the ſame Plane, BD and CE are part upon it, and part above it; which has been ſhewn to be impoſſible.

Prop. 9. Every *Triangle may be conceiv'd as contain'd in* 9
a Plane.

This is a Conſequence of the preceeding Propoſition.

Prop. 10. The *common Section or meeting of two Planes, is a* 10
right Line. Eucl. 11. Prop. 3.

X and Z, interſect. The Ends of their Section are the Points E and F, between which there may be drawn a Line upon Z and one upon X. If theſe two Lines are not one and the ſame Line, then there may be drawn between the ſame two Points more than one right Line; which is impoſſible. † therefore the Section of two Planes can be no other than that of a right Line. See P. B. *Plate* 9. *fig.* 36. *or* 37.

Prop. 11. A *right Line ſuch as AB raiſed upon a Plane X,* 11
ought to be looked upon as a Perpendicular when its End is equally diſtant from the Circumference of a Circle; deſcribed about its End A, as a Center, and when it is not, it cannot be ſaid to be Perpendicular.

This is agreeable to the Nature of a Perpendicular Line, which inclines not more to one Side than to the other.

Prop. 12. A *Line ſuch as AC Perpendicular upon the Plane X,* 12
being conceived as carried by a direct and regular Motion according to a right Line ſuch as AB, it will make the Plane Z, which will be Perpendicular in all its parts upon the Plane X.

Prop. 13. If *the Line ED, Perpendicular upon AB, the Section* 13
of Z and X, is Perpendicular upon X, all the Plane Z is Perpendicular upon X.

---

* *B.* 1. *n.* 16. † *B.* 1. *n.* 15.

For the Plane Z may be conceived as deſcribed by the direct and uniform Motion of DE upon AB; therefore by the preceeding Propoſition, the Plane Z is Perpendicular upon the Plane X.

14 Theorem 1. Between *two Lines either Parallel or not, that are in one and the ſame Plane, or between a Line and a Point, can be conceiv'd but one and the ſame Plane, in which is contain'd every right Line that is drawn between any two Lines of that Plane.*

For if there cou'd be conceived two Planes, one would be greater or leſſer; which cannot be, ſince every Superficies that is not the leaſt between its Bounds, is not a Plane.. * they are then equal; which being ſo, they are not different: For if it be ſaid that they are placed one upon the other, as they have no thickneſs, they could make only one Plane.

15 Theor. 2. Two *Planes that agree in three Points which are not upon one and the ſame Line, agree entirely.*

1. Between two of the given Points, can be conceived only one right Line. 2. Between this right Line and the third Point given, by the preceeding Propoſitions two different Planes cannot be conceived; therefore the Part of theſe two Planes between theſe three Points is one and the ſame, conſequently if this part be prolonged, it will be only one and the ſame Plane; therefore theſe two Planes are not different.

16 Corollary. Therefore *the Poſition of a Plane depends only upon three Points, which are not upon one and the ſame Line.*

Or which is the ſame, *three Points which are not upon one and the ſame Line being given, the Plane is given.*

17 Theor. 3. If *B, the Top of the Line AB raiſed upon the Plane X, is equally diſtant from C, D, E, three Points equally diſtant from its Foot A, it is Perpendicular upon X.*

1. In the Plane X ſuppoſe a Circle at an equal Diſtance from B, paſſing thro' C, D, E, which are ſuppoſed at equal diſtances from B. 2. Let us conceive a ſecond Circle, whoſe Center is A, as paſſing thro' the three Points C, D, E which by the Hypotheſis, are alſo equally diſtant from A. Theſe two Circles are the Same. † Therefore AB is Perpendicular upon X; ‡ *which was to be proved.*

18 Problem 1. From *a Point given above as B, to let fall a Perpendicular upon the Point X.* Eucl. 11. Prop. 11. (*ſame Fig.*)

Draw the two right Lines CE and DF at pleaſure, and ſet one foot of the Compaſſes upon B, with the other take the Points C, D, E, equally diſtant, thro' which draw a Circle, § to the Center of which draw a Line from B, which will by the preceeding Theorem, be the Perpendicular required.

Theorem

---

* *Sup. n.* 2. † *B.* 1. *n.* 88. ‡ *Sup. n.* 11.

§ *B.* 1. *n.* 87.

Theor. 4. If *a Line is Perpendicular upon the Point of Section of two Lines which are upon a Plane, it is ſo likewiſe upon the whole Plane.* Eucl. 11. Prop. 4. 19

For having taken Points in theſe two Lines on each Side the interſecting Point, equally diſtant therefrom, the Top of the Perpendicular will be at an equal Diſtance from four Points which are upon the Plane; * therefore this Line will be Perpendicular upon the whole Plane. †

Theor. 5. The *right Line AB, and all other in the Plane Z drawn thro' A, the Foot of the Perpendicular AC, upon the Plane, is Perpendicular upon AC, or AC is Perpendicular upon all the right Lines of the Plane Z which paſs thro' A.* 20

From A the Foot of the Perpendicular AC, deſcribe a Circle X, and produce the Line BA ſo that it may cut X in D and B. If C, the Top of the Perpendicular AC, is not equally diſtant from D and B, then AC will not be Perpendicular upon AB; ‡ But alſo AC will not be Perpendicular upon the Plane Z; § which is contrary to the Hypotheſis. AB therefore meets AC Perpendicularly; as doth likewiſe every other Line of the Plane Z, which paſſeth thro' A.

Theor. 6. If *a right Line AC meets three other right Lines AB, AF, AG Perpendicularly at one and the ſame Point, the three Lines are in one and the ſame Plane.* Eucl. 11. Prop. 5. (*ſame Figure.*) 21

Let AB and AF be in the ſame Plane Z, and that AG is not, but upon another Plane. Let a third Plane Y be conceived to paſs thro' AC and AG, and which being prolonged will cut Z; that Line AC will be Perpendicular upon the Plane Z; ‖ and therefore alſo upon the Section Line of Z and Y, * and it is upon AG by the Suppoſition, therefore upon AC and at the ſame Point A, there will be two Perpendiculars, which cannot be. †

Prob. 2. To *raiſe a Perpendicular, upon the Point A in the Plane Z.* Eucl. 11. Prop. 12. (*ſame fig.*) 22

From the Center A deſcribe a Circle; every Line that is let fall from a Point at an equal Diſtance from the Circle, is the Perpendicular required.

Theor. 7. There *can be but one Perpendicular from the ſame Point upon the ſame Plane.* Eucl. 11. Prop. 13. 23

If there can, let us conceive that the two Lines AE and AC ſuppoſed to be Perpendiculars and raiſed upon the ſame Point A, and that X be deſcribed from A as a Center: The Points E and C the Tops of the two equal Lines AE and AC being different cannot be equally diſtant from the Circumference of the Circle X; therefore they are not both Perpendicular upon the Plane Z. ‡

Theorem

---

* *B.* 1. *n.* 42. † *Sup. n.* 11. ‡ *B.* 1. *n.* 40.

§ *Sup. n.* 11. ‖ *Sup. n.* 19.

* *Sup. n.* 20. † *B.* 1. *n.* 49. ‡ *Sup. n.* 10.

24 THEOR. 8. FROM *a Point without a Plane can be drawn but one Perpendicular upon the Plane.*

LET A be the Point given above a Plane from whence there is ſuppoſed to be drawn two Perpendiculars, *viz.* AB and AC; join their lower Ends B and C by the Line BC. The three Points A, B, C, form the Triangle ABC which may be conceived to be in one and the ſame Plane. * The Angles ABC and ACB formed by the Perpendiculars AB and AC, are right ones; then ABC hath more than two right Angles; which cannot be. †

25 COROLLARY. IF *the Plane X is Perpendicular upon the Plane Z, and that from the Point A in the Plane X, be let fall a Perpendicular upon Z, it will fall upon MN the Section of Z and X.* Eucl. 11. Prop. 38. (*ſame fig.*)

IF it be not ſo, and that a Perpendicular fall from A upon C out of the Line MN; let fall AB Perpendicularly upon MN; therefore as B is in the Section of X and Z. there is upon Y two Perpendiculars AB and AC drawn from one and the ſame Point A; which has been ſhewn to be impoſſible.

26 THEOR. 9. THE *Perpendicular Line is the Shorteſt that can be drawn upon a Plane from a Point without the ſame.*

THE given Point is A, the Line AB is Perpendicular, AC is not. It muſt be proved that AB, is ſhorter than AC. To do which join C and B by a right Line. The Triangle ABC may be conceived to be in one and the ſame Plane. ‡ Now AB Perpendicular upon BC, is ſhorter than the Oblique Line AC; § *which was to be proved.*

27 COROLLARY. THEREFORE *the diſtance between a Plane, and a Point which is out of the Plane, ought to be meaſured by a Perpendicular.*

BY reaſon the Perpendicular is the ſhorteſt Line, and that there can be drawn no more than one.

28 THEOR. 10. TWO *Lines being Perpendicular upon one and the ſame Plane, they may be conceived as being in one and the ſame Plane.*

LET us conceive a third Line BD to be drawn by the bottoms of the two Lines AB and CD which are Perpendicular upon X, upon which third Line let us conceive AB or CD as moving uniformly and always Perpendicularly, which will thereby conſtitute a Plane, ‖ *Q. E. D.*

29 THEOR. 11 *AB and CD* (ſame fig.) *Perpendiculars upon X, are parallel; and if one of the two Parallels is Perpendicular upon the Plane X, the other is ſo likewiſe.* Eucl. 11. Prop. 6 and 8.

1. LET the Line BD be drawn thro' the bottom of AB and CD, which Lines are Perpendicular upon BD; * therefore the three Lines AB, CD, BD may be in one and the ſame

---

* *Sup. n.* 9. † *B.* 2. *n.* 79. ‡ *Sup. n.* 9.
§ *B.* 1. *n.* 53. ‖ *Sup. n.* 1.
* *Sup. n.* 20.

Plane. * Therefore AB and CD being Perpendicular upon BD, they are Parallel. † 2. If AB and CD are Parallel, and that AB be Perpendicular, I ſay that CD is ſo likewiſe; for having drawn BD, the Line AB is Perpendicular upon ED; ‡ therefore CD parallel to AB, is alſo Perpendicular upon BD; § *which was to be proved.*

Theor. 12. The *Section AB of the two Planes Z and X, which are Perpendicular upon Y, is a Perpendicular upon the ſame Plane.* 30
Eucl. 11. Prop. 19.

1. This Section is a right Line, ‖ and ſeeing the Planes Z and X are Perpendicular upon Y, the Line AP inſomuch as it is conſidered in Z cannot be conceived as leaning more on one ſide thereof than the other: And by the ſame Reaſon, inſomuch as it is conſidered in X, it cannot be conceived to incline more to to one ſide of the Plane than to the other; wherefore it may be conceived by the Means of three Points in the Plane Z equally diſtant from P, which are alſo equally diſtant from A; therefore AP is Perpendicular upon the Plane Y. *

Theor. 13. The *Section of X and Z being Perpendicular upon Y, theſe two Planes and all others whoſe Section is AP, are Perpendicular upon Y.* Eucl. 11. Prop. 18. (*ſame fig.*) 31

For all theſe Planes may be conceived as generated by the Motion of AP. †

Theor. 14. If *three Points in one and the ſame Plane, and not upon one and the ſame Line, are at an equal Diſtance from a ſecond Plane, theſe two Planes are parallel.* 32

The Poſition of a Plane depends only upon three Points; ‡ conſequently if three Points of a Plane are equally diſtant from a ſecond Plane, all other Points of theſe two Planes are at an equal diſtance; therefore they are Parallel.

Theor. 14. The *Planes X and Z being Parallel, if the Line AB is Perpendicular upon X, it is ſo alſo upon Z. And if AB is Perpendicular upon X and upon Z, the two Planes are Parallel.* 33
Eucl. 11. Prop. 14.

If it is pretended that AB is Perpendicular upon X and not ſo upon Z, let us ſuppoſe a Perpendicular at the Point A to be raiſed towards Z ſuch as AC, which is ſhorter than AB. § From C conceive a Perpendicular upon X, which is alſo ſhorter than AC, therefore yet more ſhorter than AB; hence the Point D approaches nearer Z than to A; therefore Z and X not being at an equal diſtance, they are not Parallel; which is contrary to the Hypotheſis. Therefore a Line which is Perpendicular upon one of the Planes, is ſo upon the other. *The reſt is eaſy.*

Theor. 16. The *Sections AB and CD of two Parallel Planes cut by a third Plane, are Parallel Lines.* Eucl. 11. Prop. 16. 34

---

* *Sup. n.* 28. † *B.* 1. *n.* 68. ‡ *Sup. n.* 20. § *B.* 1. *n.* 69. ‖ *Sup. n.* 10.

* *Sup. n.* 17. † *Sup. n.* 12. ‡ *Sup. n.* 16. § *B.* 1. *n.* 53.

The Sections AB and CD are right Lines, * which are in the third Plane, from whence being prolonged, they would meet if they were not Parallel; conſequently the other two Planes in which they are contain'd being prolonged, would alſo meet, therefore are not Parallel, contrary to the Suppoſition; therefore it cannot be ſaid, that AB and CD are not Parallel.

35 Theor. 17. Right *Lines which are Parallel to one and the ſame right Line, altho' upon different Planes, are Parallel one to another.* Eucl. 11. Prop. 9.

Let CE be Parallel to AB, to which DF in another Plane is alſo Parallel; I ſay that CE and DF are parallel to each other. From the Point B upon AB raiſe BC and BD Perpendicularly, croſſing CE, and DF, at the Points C and D. By means of the three Points B, C, D, may be conceived a Plane; † AB Perpendicular upon BD, and BC, is Perpendicular upon that Plane, ‡ but CE and DF being ſuppoſed parallel to AB, are alſo Perpendicular upon the ſaid Plane and Parallel to each other; § *which was to be proved.*

36 Theor. 18. All *parallel Lines in the Plane X, met by other parallel Lines in the Plane Z, form Angles equal one to another.* Eucl. 11. Prop. 10. (*ſame fig.*)

BD and NO are Parallel to each other upon the Plane X; and BC, and MN upon the Plane Z: It muſt be proved that the Angle CBD is equal to MNO. To do which, draw DF and CE Parallel to AB. Since that Parallels between Parallels are equal, for they conſtitute the ſame Angles, ‖ conſequently equal ones; * I ſay, ſince CE is Parallel to DF by the preceeding Theorem; therefore BD=NO, and BC=NM, and CD=MO; therefore the Triangles CBD and MNO are equal and ſimilar, therefore the Angle CBD is equal to MNO; *Q. E. D.*

37 Theor. 19. Two *Lines AB and AC which meet at the Point A, and are Parallel to two other Lines ED and DF which meet at D, if they are not upon one and the ſame Plane, the Planes BC and EF, are Parallel.* Eucl. 11. Prop. 15.

From A let there be drawn a Perpendicular upon the Plane EF which it meets at the Point G, from which draw GH parallel to DE, and GI parallel to DF: Then by Theorem 17, † AB and GH are Parallel, as alſo AC and GI: Hence AG being *by conſtruction*, Perpendicular upon GI and upon GH, it is ſo likewiſe upon AC and upon AB, ‡ and therefore upon the Plane BAC; § hence the two Planes muſt needs be Parallel; for if they were not, they (*being produced*) would meet in ſome Point or other, from whence might be drawn right Lines to the Points A and G, which therefore are not Parallel, contrary to what has been demonſtrated, ſince AG being Perpendicular upon the ſaid Planes, it ought to be ſo upon all the Lines drawn upon them thro' the Points A and G; * which cannot be if thoſe Lines cou'd meet.

---

* *Sup. n.* 10. † *Sup. n.* 9. ‡ *Sup. n.* 19. § *Sup. n.* 29. ‖ *B.* 2, *n.* 27.

* *B.* 2. *n.* 110. † *Sup. n.* 35. ‡ *B.* 1. *n.* 69. § *Sup. n.* 19.

38 THEOR. 20. TWO *right Lines being cut by parallel Planes, are divided Proportionally.* Eucl. 11. Prop. 17.

LET the two right Lines be AB, CD, cut by three parallel Planes X, Y, Z, at the Points A, L, B, D, M, C; I ſay that they are divided Proportionally, that is, AL. LB::CM. MD. For upon each of the extreme Planes X and Z, the Points of Section are joined by the right Lines AC and BD, and the Diagonal AD being drawn interſects the Plane Y at the Point N, from which are drawn the Lines LN and MN thro' the Points of Section L and M. The Triangle ABD may be conceived in one Plane, † as may alſo that of ADC in another, if it is not the Same; each of the Planes of theſe Triangles being cut by parallel Planes, the Section Lines are Parallel, ‡ and the Triangles are alſo cut Parallel to their Baſes, and therefore AL. LB::AN. ND::CM. MD, § and conſequently AL. LB::CM. MD; ‖ *which was to be proved.* If there had been more Planes and more Lines, it would have been the ſame thing.

SECT. II.

OF the Compoſition of Solids according to their Surfaces, and Solidity.

DEFINITION I.

39 WHEN *three or more plane Angles, which are upon different Planes, (or which have not one and the ſame baſe) meet at their Vertex, the Angle which they conſtitute is called a ſolid Angle.*

Two plain Angles only cannot form a Solid Angle; it requires at leaſt three plane Angles, which croſs or meet about a Point, like a well cut Diamond.

40 DEF. 2. SOLIDS *bounded by rectilinear Surfaces, or parallel Planes, are called* Parallelopipedons. P. B. Plate 7. fig. 20.

41 DEF. 3. SOLIDS *contained between Planes all ſimilar and equal, are called* Polyedrons. *When the Figures which bound them are regular, and all their Angles equal, they are called regular Polyedrons.*

4 DEF. 4. A *regular Solid receives a different and particular Name, according to the Number of its planes or Surfaces.*

U 1.

---

* *Sup. n.* 20. † *Sup. n.* 9. ‡ *Sup. n.* 134.

§ *B.* 4. *n.* 16. ‖ *B.* 3. *n.* 53.

1. **When** *it is bounded by four equal and Equilateral Triangles, it is called a,* Tetraedron. *See* P. B. *Plate* 1. *fig.* 3.

2. **Hexaedron,** *or* Cube, *when it is bounded by ſix equal Squares, See* P. B. Plate 1. *fig.* 1.

3. **Octoedron,** *when bounded by eight equal and Equilateral Triangles.* P. B. Plate 1. fig. 4.

4. **Dodecaedron,** *when it's bounded by twelve equal and equilateral Pentagons.* P. B. Plate 1. fig. 6.

5. **Icosaedron,** *when bounded by* 20 *equal and equilateral Triangles.* P. B. Plate 1. fig. 5.

43 Def. 5. A *Line whoſe Top is unmoveable, and its bottom made to move round a Figure.* 1. *If the Figure be a Polygon, the Line will deſcribe a Solid, which is called a* Pyramid, *ſuch as ABCD.* 2. *If the Figure about which the bottom of the Line moves is a Circle, the Solid generated thereby is a Cone, as X.* P. B. Plate 7. fig. 28. and Plate 8. fig. 31.

In theſe Figures the Line drawn from the Vertex to the Center of the Baſe, is called its *Axis,* and the Solid is called either Right or Oblique, according as its *Axis* makes either a right or an oblique Angle with the Baſe.

3. If the Top of the Line, which deſcribes the Solid is not unmoveable, but that whilſt its bottom moves round the Baſe, its top moves round a Similar, equal Figure, and alike poſited, ſuch as ABCDEF, it is a *Priſm.* P. B. *Plate* 7. *fig.* 19, 21, 22, *&c.*

4. If the Baſe is a Circle, it will be a *Cylinder,* ſuch as *ABXZ.*

44 Def. 6. If *a Circle revolves about its Diameter, it deſcribes a* Sphere, *the Diameter of which is called the* Axis *thereof; the Center of the Circle, whoſe revolution makes the Sphere, is the Center of the Sphere; the Lines drawn from the Center to the Circumference are the Radii; thoſe which paſs thro' the Center and terminate at the Circumference are Diameters.*

45 Def. 7. A *regular Polygon revolving about a right Line which paſſeth thro' its Center, deſcribes what is called a* Spheroid, *that is, a different Sort or Species of a Sphere, ſuch as is X and Z.*

46 Def. 8. A *Solid A is ſaid to be* Circumſcribed *about another Solid B which it contains, if it is the leaſt of all ſimilar Solids capable of containing B; or rather, if B is the greateſt of all ſimilar Solids which A is capable of containing.*

47 Def. 9. A *Solid B is ſaid to be* inſcribed *in another Solid A in which it is contained, if it is the greateſt of all ſimilar Solids that can be contained in A; or which is the Same, if A is the leaſt of all ſimilar Solids that can contain it.*

48 Def. 10. A *regular Body is that which is bounded by regular, Similar, and equal Figures, whoſe ſolid Angles are alſo all equal.*

It will be hereafter demonſtrated, that there are only five regular Bodies.

Theorem 1. If *a ſolid Angle is comprized under three plane Angles, any two of theſe plane Angles taken together, are greater than the Third.* Eucl. 11. Prop. 20. 49

The three plane Angles BAD, BAC, CAD make a Solid Angle; it muſt be proved that two of theſe plane Angles taken together are greater than the Third; as for Example BAD < BAC+CAD, there is no leſſer Plane than BAD, between the Lines AB and AD; * therefore the rectilinear Plane BAD is leſſer than the Curvilinear one ABCD. In the ſame manner it is demonſtrable, that the plane Angle BAC is leſſer than BAD and CAD together.

Corollary. The *Sides of three plane Angles, which make a ſolid Angle, being taken equal, the Baſes of theſe three plane Angles, make a Triangle.* Eucl. 11. Prop. 22. (*ſame fig.*) 50

Let the three Lines AB, AC, AD, Sides of three plane Angles, which make the ſolid Angle A, be taken equal. It muſt be demonſtrated that the three Lines BC, CD, DB. Baſes of the three plane Angles BAC, CAD, DAB, make a Triangle. To do which it is enough that two of theſe Lines taken together, are longer than the Third. † Now this is ſo, for the two Angles of which theſe Lines are the Baſes, are greater than the Angle which is upon the Third; and conſequently this third Line is ſhorter. ‡

Theor. 2. All *the plain Angles which comprize a ſolid Angle, are together leſs than four right Angles.* Eucl. 11. Prop. 21. 51

1. Let us begin with the ſolid Angle A, compoſed of three Planes: by the preceeding Theorem the Angles BCA+DCA are greater than the Angle BCD, and likewiſe ADC+ADB greater than CDB; as alſo ABD+ABC greater than DBC. Hence the ſix Angles at the Baſes of the three Triangles which form the ſolid Angle A, are greater than the three Angles of the Triangle BCD, *that is*, greater than two right Angles. § Now all the Angles of the three Triangles which form the ſaid ſolid Angle A, are together equal to ſix right ones. ‖ If then from that Sum be taken more than two right ones, the Content of the ſix Angles at the Baſe, there will remain leſs than four right Angles for the Content of the ſolid Angle A; *which was to be proved.* 2. By the ſame Method it is ſhewn that the ſolid Angles formed by four Planes, are leſs than four right ones. 3. If X is a ſolid Angle comprehended under five Triangles, whoſe angular Point ought to be conceived as above the Plane, the preceeding Theorem will alſo prove that the Angle DBC is leſſer than the two Angles DBA and CBA, and the Angle BCE is leſſer than the Angles ACB and ACE taken together, the ſame of the others. But alſo all the Angles of the Polygon BCEFD,

---

* *Sup. n.* 3. † *B.* 2. *n*, 67. ‡ *B.* 2. *n.* 104.
§ *B.* 2. *n.* 75. ‖ *B.* 2. *n.* 75.

the Baſe of the ſolid Angle, are equal to ſix right ones; * hence all the Angles at the Baſe of the five Triangles which make the ſolid Angle X, are greater than ſix right ones, ſeeing they are greater than the Angles of the Polygon, as was ſhewn before. All the Angles of the five Triangles which make the ſolid gle are equal to ten right ones; therefore ſince thoſe of their Baſes are more than ſix right ones, thoſe of their Vertices are leſs than four right ones. *Q. E. D.*

This *Theorem may be demonſtrated thus.*

1. Let us conceive A to be a Point in a Plane, and the Vertex of ſeveral Triangles whoſe Baſes are the Sides of X. All the Angles about A make only four right Angles. 2. If the Point A be raiſed up, then the vertical Angles of the Triangles will become leſs, having the ſame Baſes and longer Sides; † this being ſo, all the Angles will together be leſs than four right angles.

52 Problem 1. Having *three plane Triangles, two of which taken together are greater than the Third, but which all together are leſs than four right Angles, to make therewith a ſolid Angle.* Eucl. 11. Prop. 23.

Let the three plane Triangles be A, B, C, 'tis required only to join them in one and the ſame Point, ſo that the containing Sides may coincide together, *that is,* that AE unite with BE, BF with CF, and CG with AD; this will form the ſolid Angle required, according to the Definition thereof. ‡

53 Prob. 2. To *make a ſolid Angle upon a given right Line, and at a Point given in that Line, equal to a ſolid Angle given.* Eucl. 11 Prop. 26.

Let the given Line be AD, and A a given Point in that Line, required to make at that Point, a ſolid Angle, equal to a ſolid Angle given. Let the three Planes of the given Angle be DAE, EBF, FCG; having made three other Planes equal thereto, and joined them together at the Point A in the Manner as was explained in the preceeding Problem, it is evident that they conſtitute the ſolid Angle required.

54 Theor. 3. If *there are two equal plane Angles, thro' the Vertices whereof are drawn two Lines, raiſed upwards, and containing equal Angles with the Lines firſt given, each to each, and from a Point taken in each of thoſe elevated Lines, be drawn lines Perpendicular to the Planes, which contain the Angles firſt given, and from the Points were thoſe Penpendiculars fall, be drawn right Lines to the Vertices of the Angles firſt given, the Angles which thoſe Lines make with thoſe elevated Lines, are equal one to another.* Eucl. 11. Prop. 35.

In Euclid this Propoſition is uſed for the Demonſtration of other following ones, but as the Method of this Treatiſe has no need of it, I therefore paſs it without taking any farther Notice thereof.

---

* *B. 2. n.* 123. † *B. 2. n.* 99. ‡ *Sup. n.* 39.

THEOR. 4. THERE *cannot be more than five different regular Bodies.* 55

THAT IS five different Solids comprized under plane regular Figures all equal and ſimilar, and whoſe Angles are all equal. In a Sphere, the whole is equal; but it is different in a Solid comprized under Planes. It has been proved that all the plane Angles, which form a ſolid Angle, are leſs than four right Angles. Let us now ſee which are the Plane, Similar, Equal, regular Figures that can make a ſolid Angle. 1. Three equal and equilateral Triangles may make a ſolid Angle, for the three Angles of the Triangles which contain a ſolid Angle, make only three times ſixty Degrees, each Angle of an equilateral Triangle being ſixty: Three of theſe Triangles joined together may therefore make a ſolid Angle, ſuch as are thoſe of the Tetraedron. 2. Four of theſe Triangles may alſo make a ſolid Angle; for the four Angles which they contain are only four times Sixty; which is leſs than four right ones. Four of theſe Triangles make the Angle of the Octoedron. 3. Five of theſe Triangles may alſo compoſe a ſolid Angle, for the Angles of the Planes which they comprize, make only five times ſixty Degrees; which is leſs than four right Angles. The Angle of the Icoſaedron is made by five of theſe Triangles.

Six equilateral Triangles cannot make a ſolid Angle; for the ſix plane Angles which they contain, make four right Angles, 6 times 60=360 the Sum of four right Angles; therefore theſe ſix Triangles make a plane Angle, and not a Solid one. 4. Three Angles of a Square may make a ſolid Angle; for their Sum is leſs than four right Angles. The Angle of the Cube is compoſed of three Angles of the Square. No other Solid can be conceived as compoſed of Squares for four cannot make a ſolid Angle; much leſs, either five or ſix. 5. Three Pentagons form a ſolid Angle; for their Angles make only 324 Degrees, which is leſs than four right Angles. Each angle of the Dodecaedron is contained under three Pentagons, more than three Pentagons cannot make a ſolid Angle; for four make 432 Degrees, which is above the Sum of four right Angles. Three Hexagons cannot conſtitute a ſolid Angle; for each being 120 Degrees, the three make 360, which is the Sum of four right Angles: Therefore they cannot make a ſolid Angle. † The more Sides a Polygon has the greater is the Angle thereof; therefore if three Hexagon's cannot make a ſolid Angle, there is no poſſibility that three Heptagon's ſhould do it; hence there is only the *Equilateral Triangle*, *Square* and *Pentagon* that can do it: But a ſolid Angle may be made of three, four, and five equilateral Triangles, there cannot therefore be any more than five different regular Bodies.

---

† *Sup. n.* 5.

Evident Propoſitions.

Proposition I.

56 A *Figure is greater than that about which it is circumſcribed, and leſſer than that in which it is inſcribed.*

57 Prop. 2. Two *Priſms having the ſame Height, that which hath the leaſt Baſe, (and which may conſequently be contained in the other) is the leaſt.*

For it is evident that it is contained therein. Now that which contains, is greater than that which is contained.

58 Prop. 3. Of *two Priſms circumſcribed about a Cylinder, that which is moſt like the Cylinder, has the moſt Sides.*

The Baſe of the propoſed Cylinder is X; let that of the Priſm which hath feweſt Sides, and which is circumſcribed about the Cylinder, be called Y, and Z that of another Priſm having more Sides. and circumſcribed about the ſame Cylinder. The Polygon Z is leſſer than the Polygon Y. * The Priſms, whoſe Baſes are Polygons are of the ſame Height, being Circumſcribed about one and the ſame Cylinder, therefore that which is upon Z, and which hath therefore moſt Sides, is leſſer than that which hath fewer, and whoſe Baſe is Y; † it therefore approaches neareſt to the Cylinder; *Q. E. D.*

59 Prop. 4. Therefore *a Priſm whoſe Sides are infinite, is ſo near like a Cylinder, that there is no perceivable difference between them, the Cylinder may therefore be ſuppoſed to be a Priſm having an infinite Number of Sides.*

60 Prop. 5. Of *two Priſms inſcribed in a Cylinder, that which approaches neareſt to a Cylinder, hath moſt Sides.*

The Baſe of the propoſed Cylinder is X, the two inſcribed Polygons Z and Y are the Baſes of two Priſms inſcribed in the Cylinder whoſe Baſe is X. The Polygon Y which hath moſt Sides is greateſt, and approaches nearer to a Circle than the Polygon Z. ‡ The two Priſms, whoſe Baſes are Z and Y, are of the ſame Height, being inſcribed in one and the ſame Cylinder; therefore the Priſm which is upon Y is greater than that which is upon Z. § It therefore approaches neareſt to a Cylinder; *Q. E. D.*

61 Prop. 6. Two *Pyramids having the ſame Height, that is the greateſt which hath the longeſt Baſe.*

For it is evident that if one be conceived as put upon the other, that which hath the greateſt Baſe will contain that which hath a leſſer.

62 Prop. 7. Of *two Pyramids circumſcribed about a Cone, that which hath moſt Sides moſt reſembles a Cone.*

---

* *B.* 2. *n.* 149. † *Sup. n.* 57. ‡ *B.* 2. *n.* 151.
§ *Sup. n.* 56.

PROP. 8. OF *two Pyramids inſcribed in a Cone, that which hath moſt Sides approaches the neareſt to a Cone.* 63

PROP. 9. A *Cone may therefore be ſuppoſed to be a Pyramid of infinite Sides.* 64

PROP. 10. THE *more ſides a Polygon hath, the nearer the Spheroid which it forms approaches to its inſcribed Sphere.* 65

FOR the more Sides a Polygon (*which may be called the Generator of a Spheriod*) hath, the nearer it approaches to a Circle; therefore the Spheroid which it deſcribes by its revolution approaches nearer its inſcribed Sphere. *Q. E. D.*

PROP. 11. THEREFORE *a Sphere may be taken for a Spheroid formed by a Polygon of an infinite Number of Sides.* 66

SECT. III.

OF the Surfaces of Solids.

THEOREM I.

THE *Surface of a right-angled Priſm is equal to a Parallelogram having the ſame Height, and its Baſe equal to the Circumference of the Priſm.* 67

THE Superficies of a right angled Priſm are Parallelograms having all the ſame Height, whoſe Baſes taken together make the Circumference of the Priſm: It is therefore equal to a Parallelogram having the ſame Height as the Priſm, and whoſe Baſe is equal to its Circumference; *which is evident.* The Surfaces of the Baſes of the Priſm are excluded.

COROLLARY I. THEREFORE *ſeeing that a right Cylinder may be taken for a Priſm of infinite Sides,* * *its Surface is equal to a Parallelogram of the ſame Height and its Baſe equal to the Circumference of the Circle, which is the Baſe of the Cylinder.* 68

COROL. 2. THEREFORE *all that has been demonſtrated concerning the Ratio which Parallelograms have one to another, agrees to Cylindrical Surfaces.* 69

1. THE Ratio which the Surfaces of two Cylinders bear one to another is compounded of their Heights, and of the Circumference of their Baſes. 2. In two Cylinders, if it be as Height is to Height ſo is Baſe to Baſe, *that is,* if the Ratio of their Surfaces is compounded of two equal Ratio's, the Ratio is duple. † 3. If two Cylinders have either their Heights or their Baſes equal, their Surfaces are one to another as their parts which are unequal. ‡

Theorem

---

* *Sup. n.* 59. † *B.* 4. *n.* 71. ‡ *B.* 4. *n.* 75.

70 THEOR. 2. *IF a Priſm hath a Polygon for its Baſe, and the Apotome thereof for its Height, its Surface is double that of the Polygon which is its Baſe.*

LET Z be a Priſm, whoſe Baſe is the Polygon X. The Apotome of X is AG, *that is*, a Perpendicular drawn from its Center A upon one of its Sides BC; It muſt be demonſtrated that if FG, the Height of Z is equal to AG the Apotome of X, the Surface of Z is double that of X. Having drawn Lines from each Angle of the Polygon X to the Center A, they make as many Triangles as Z hath Sides or Faces, which Faces are Parallelograms. Now theſe Parallelograms as BCED, and the Triangles as ABC, have the ſame Height and Baſe. Therefore the Parallelograms are double the Triangles, * and conſequently the Surface of Z compounded of the Parallelograms is double that of X, equal to all the Triangles.

71 COROLLARY 1. THEREFORE *if the Height of a Cylinder (which may be eſteemed as a Priſm) is equal to the Radius of the Circle which is its Baſe, its Surface is double that of the Circle.*

72 COROL. 2. THEREFORE *if the Height of a Cylinder be twice the Radius, or once the Diameter of the Circle which is its Baſe, its Surface is Quadruple that of the Circle.*

73 THEOR. 3. THE *Surface of the Circumference of a Cylinder is equal to that of a Circle, whoſe Radius is a mean Proportional between the Height of the Cylinder and the Diameter of the Circle which is its Baſe.* Archimedes, 1. Prop. 16.

X is a Cylinder, whoſe Height is AB, the Circuit of its Baſe which is the Circle Z, is BD, whoſe Circumference is FG or BD, and BC, the Double of the Circuit. The Surface of X is equal to the Parallelogram ABD or to the Triangle ABC, as is that of the Circle Z to the Triangle EFG. † I ſuppoſe KH, the Radius of the Circle Y, to be a mean Proportional between the Height of X, and 2 EF the Diameter of Z. Let KL be the Circuit of Y, hence its Surface is equal to the Triangle HKL; it is therefore required only to prove that the Triangle HKL is equal to the Triangle ABC: For all Circles have ſame the Ratio between the Radius and Circumferences.

By the Hypotheſis ∺ AB. HK. 2EF. Now HK. KL::2EF. 2FG or BC; and *Alternately* HK. 2EF::KL. BC; then ſeeing that AB. HK::HK. 2EF::KL. BC. Hence AB. HK::KL. BC, ‡ wherefore AB×BC=HK×KL. § Now the Triangles ABC and HKL are the Halves of the Rectangles; therefore they are equal; *which was to be proved.*

74 THEOR. 4. THE *Surface of a Pyramid is equal to a Triangle, whoſe Height is equal to the Height of each of its Sides, and whoſe Baſe is equal to the Circuit of the Baſe of the Pyramid, or to a Parallelo-*

---

* *B.* 2. *n.* 133. † *B.* 2. *n.* 154. ‡ *B.* 3. *n.* 53.
§ *B.* 3. *n.* 56.

Parallelogram having the ſame Height, and but half the Baſe. Archimedes 1. Prop. 10 and 11.

Each Side of the Pyramid is a Triangle; theſe Sides being equal, the Triangles are equal one to another, and to a right Angled Triangle having the ſame Height, and its Baſe equal to all the Baſes of theſe Triangles taken together. * Now a Triangle is equal to a Parallelogram having the ſame Height, and half the Baſe, † *which was to be proved.*

75 Corollary 1. Therefore *ſeeing that Cones may be conſidered as Pyramids, the Surface of a Cone is equal to a right angled Triangle having the ſame Height as the Side of the Cone, and its Baſe equal to the Circuit of the Baſe of the Cone, or to a right angled Parallelogram having the ſame Height, and whoſe Baſe is equal to half the Baſe of the Cone.*

The ſuperficial Height of a Cone and of a Pyramid is the ſhorteſt right Line that can be imagined poſſible to be drawn upon the Surface, from the Vertex down to its Baſe.

76 Coroll. 2. Therefore *all that has been demonſtrated of the Ratio's and proportions between ſeveral ſimilar Rectangles, agrees to the Superficies of Cones.*

1. The Surfaces of Cones are one to another in a Ratio compounded of their Height and Baſe. 2. If Height is to Height as Baſe to Baſe, the Surfaces are in duple Ratio. 3. If two Cones have equal Heights or equal Baſes, their Surfaces are one to another as their parts which are unequal. 4. Therefore ſeeing that the Circumferences of Circles are one to another as their Diameters, the Surfaces of two Cones having the ſame Height, are one to another as the Diameters of their Baſes. 5. A Rectangle being given, there may be found one or more ſimilar ones having therewith, any Ratio you pleaſe; alſo a Cone being given there may be found one or more other ſimilar Cones bearing to the given one any Ratio required.

77 Theor. 5. The *Surface of a Cone X is to that of the Circle BCD, which is its Baſe, as AB the Height of its Surface is to the Radius of the Circle.* Archimedes 1. Prop. 18.

The Surface of the Circle is equal to the right angled Triangle Z whoſe ſide D is equal to the Radius BC, and the ſide E to the Circumference of the Circle. ‡ The Surface of the Cone X is equal to the rectangled Triangle Y, whoſe ſide F is equal to AB, and the ſide G to the Circuit of its Baſe. § G and E being each equal to the Circumference of the Circle, which is the Baſe of the Cone, they are equal one to another; wherefore the Surfaces of theſe two right angled Triangles Y and Z, which have equal Baſes, *viz.* G and E, are one to another as F to D; ‖ but AB=F, and BC=D: therefore X the Surface of the Cone, is to that of the Circle which is its Baſe, as AB the

---

* *B.* 2. *n.* 143. † *B.* 2. *n.* 133. ‡ *B.* 2. *n.* 154. § *Sup. n.* 75. ‖ *B.* 4. *n.* 75.

height of its Surface is to BC the Radius of the Circle which is its Baſe; *which was to be proved.*

78 **Theor. 6.** The *Surface of a Circle whoſe Radius is a mean Proportional between the Height of the Cone, and the Radius of its Baſe, is equal to the Cone's Surface.* Archimedes 1. Prop. 17.

Let X be a Cone, the height of whoſe Surface is AB, and the Circuit of its Baſe BD; hence its Surface is equal to the right angled Triangle ABD. The Line BC is the Radius of its Baſe. Suppoſe EF to be a mean Proportional between AB and BC, and alſo the Radius of a Circle whoſe circuit is FG; hence the Surface of the Circle is equal to the Triangle EFG: Conſequently 'tis required only to prove that ABD=EFG. By the Hypotheſis AB. EF::EF. BC, and ſeeing that the Circumferences of Circles are one to another as their Diameters, EF. BC ::FG, BD. Hence AB. EF::EF. BC::FG. BD. then AB. EF:: FG. BD; therefore AB×BD=EF×FG. * Now ABD is the half of AB×BD, † as alſo EFG is the half of EF ×FG. Thoſe Triangles are therefore equal; *which was to be proved.*

79 **Theor. 7.** If *the Height and Baſe of a Priſm are equal to the Height and Baſe of a right angled Pyramid, the Surface of the Priſm is double that of the Pyramid.*

Each ſide of the Priſm is a Parallelogram, and each ſide of the Pyramid is a Triangle. In the Caſe propoſed theſe Parallelograms and Triangles are of the ſame Height, and the ſame Baſe; the Parallelograms are therefore double the Triangles; ‡ therefore the Surface of the Priſm is double that of the Pyramid; *which was to be proved.*

80 **Corollary.** Therefore *ſeeing Cones may be eſteemed as Pyramids, and Cylinders as Priſms, the Surface of the Cylinder may be ſaid to be double that of the Cone having the ſame Height and Baſe.*

Take notice, that the real Height of a Pyramid, muſt not be confounded with that of the plane Surfaces which form it; the Firſt is a Line drawn from the top of the Pyramid perpendicularly upon the Plane which is its Baſe; and the Second a different Line drawn alſo perpendicularly from the Vertex of the Pyramid, but upon one of the Sides of the Polygon which is its Baſe, which ſecond Line is longer than the Firſt. It is the ſame of Cones, whoſe real Heights or Axes are different from thoſe of their ſides.

81 **Lemma 1.** The *Surface of BCDE, the Fruſtum of the Cone ADE, is equal to that of the Trapezia GHLK.*

The Surface of the Cone AED is equal to the right angled Triangle FGH of which FG=AD, and GH equal to the Circumference of the Circle DE, § and that of the Cone ABC to the right angled Triangle FKL of which FK=AB, and KL to

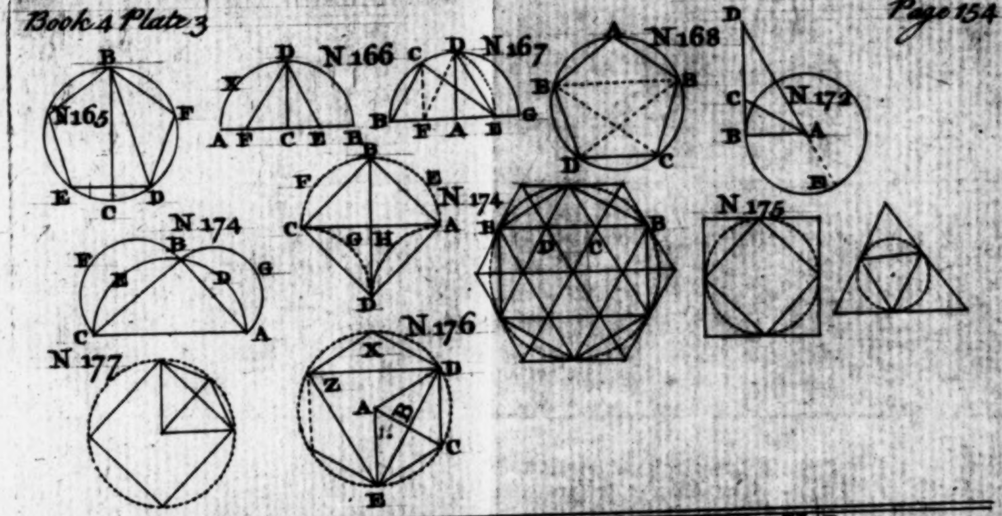
---

* *B.* 3. *n*, 56. † *B.* 2. *n.* 133. ‡ *B.* 2. *n.* 133.

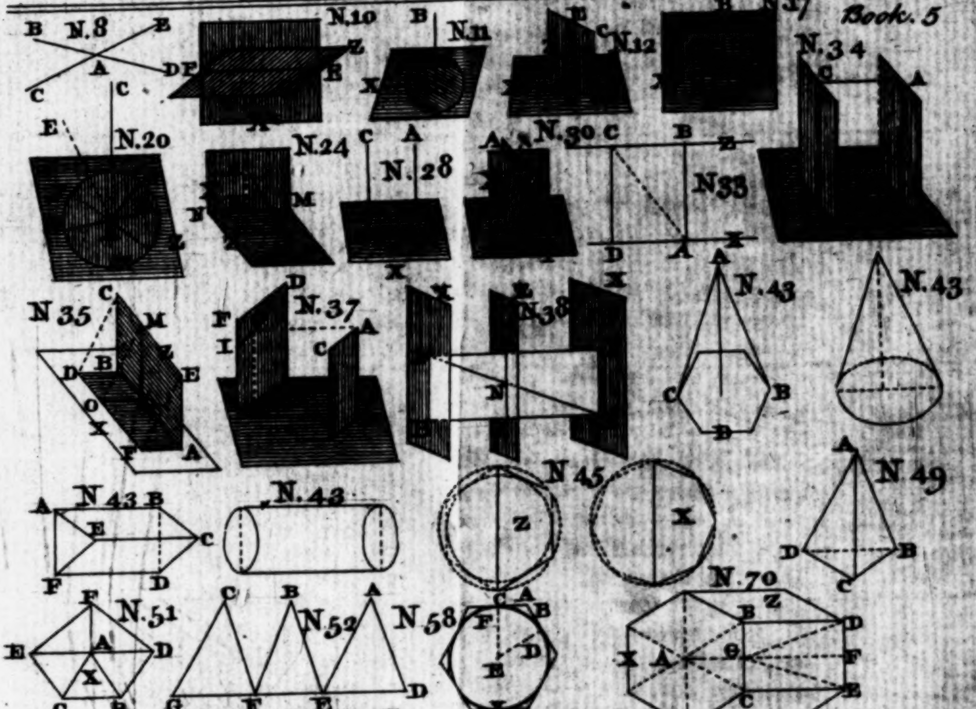
§ *Sup. n.* 74.

Book 4 Plate 3

Page 154



Book. 5



to the Circumference of the Circle CB: Then taking away FKL from FGH, the Remainder GHLK is equal to the Fruſtum BCDE, *which is evident.*

Lemma 2. HC *and KL are the Radii of the two Circles of the two Baſes of the Fruſtum of a Cone. KC is divided into two equal parts at M, and MN is parallel to KL and HC: I ſay that the Surface of the Fruſtum is equal to the Rectangle made of KC, the Height of that Surface, and of the Circumference of a Circle, whoſe Radius is MN.* 82

For the Figure CKLH=OCKP, becauſe of the equality of the two Triangles HON and NPL, added or ſubtracted, but the Figure CKLH is to that which has been proved equal to the Fruſtum of the Cone, in a Ratio of the Radius of the Circle to its Circumference *. And OCKP is alſo in the ſame Ratio to the Rectangle made of KC by the Circumference of the Circle whoſe Radius is MN; therefore the Rectangle is equal to the Fruſtum; † *Q. E. D.*

Lemma 3. If *the Angles of a Spheroid as X, are joined by Planes perpendicular to its Axis, which divide it into ſeveral parts, the Parts are either Cones as C, or Fruſtums of a Cone as B, or Cylinders as A.* Which is evident. 83

Theor. 8. Each *Surface of the Portions of a Spheroid is equal to the Rectangle made of its correſponding part of the Axis, and Circumference of the Circle or Sphere inſcribed in the Spheroid.* 84

As to the Part A, it is very eaſy, ſeeing that 'tis a Cylinder whoſe Surface is equal to the Rectangle of EF by the Circumference of a Circle whoſe Diameter is EH, ‡ which is equal to the Diameter of the Circle or Sphere inſcribed in the Spheroid X; *which was to be proved.*

Concerning the Part B, it muſt be demonſtrated that the Surface of the Part B (*which is the Fruſtum of a Cone*) is equal to a Rectangle made of KL the correſponding Part of the Spheroid's Axis, and of the Circumference of the Circle inſcribed in the ſaid Spheroid, whoſe Diameter is CN. Divide the Height KL into two equal Parts, by drawing CM parallel to FK and to EL: Alſo draw GE parallel to KL which is equal thereto. The Triangles EFG and ACD are right Angles; therefore GFE and GEF make a right Angle, the Angle GFE being therefore equal to FCD, § the Angle FCD ſubtracted from the right Angle FCA, the Remainder DCA is equal to GEF, hence the two Triangles ACD and EFG are equiangular: Then GE or KL. EF::CD. CA, ‖ wherefore KL is to EF as the double of CD, which is GM, is to the double of AC, which is CN. Let CM be the Diameter of a Circle whoſe Circumference is Y; and CN that of a Circle whoſe Circumference is Z. Then CM. CN::Y. Z. Then

---

* *B* 4. *n.* 75. † *B.* 3. *n.* 52. ‡ *Sup. n.* 68.

§ *B.* 2. *n.* 25. † *B.* 4. *n.* 10.

KL. EF::Y. Z. Then $KL\times Z=EF\times Y$. * Now the Surface of the Fruſtum of the Cone B, is equal to $EF\times Y$. † Therefore that Surface equal to $EF\times Y$, is equal to a Rectangle made of KL by Z, the Circumference of a Circle whoſe Diameter is CN; *which was to be proved.*

For the Part C. From O the Middle of BD the Side of the Cone C, draw to A, the Center of the Circle inſcribed in the Spheroid, the Line AO; and thro' D a parallel to AO; hence as BG is the half of BD, the half of DF is AO; therefore DF is the Diameter of the Circle whoſe Radius is AO. The Surface of the Cone C is equal to a right angled Triangle whoſe height is BD, and the Baſe a Circle whoſe Diameter is DG, ‡ or to a right angled Parallelogram, having the ſame height BD, and whoſe Baſe is half that of a Circle whoſe Diameter is DG, § or which is the ſame, whoſe Baſe is equal to a Circle having for its Diameter DE. Let the Circumference of the Circle be called Y, and Z that of the Circle whoſe Diameter is DF: It muſt be proved that $BD\times Y=BE\times Z$. The two Triangles DEF and DEB are Similar: ‖ therefore BD. BE::DF. DE. Now DF. DE::Z. Y: * Then BD. BE::Z. Y: † therefore $BD\times Y=BE\times Z$; *which was to be proved.*

85 THEOR. 9. THE *Surface of a Spheroid is equal to the Rectangle made of its Axis, by the Circumference of the Circle of the inſcribed Sphere.*

BY the preceeding Theorem, ſeeing that the Surface of each part of the Spheroid is equal to the Rectangle, made of each correſponding part of its Axis, and the Circumference of the Circle of the inſcribed Sphere, all the whole Surface is equal to the Rectangle of the whole Axis by the Circumference of the Circle of the Sphere inſcribed, ſince that the whole and its parts when multiplied by one and the ſame Quantity give equal products. ‡

86 THEOR. 10. THE *Surface of a Sphere is equal to the Rectangle of its Axis, and the Circumference of a Circle having the ſame Diameter as that Sphere.*

THE Sphere may be eſteemed as a Spheroid formed by a regular Polygon having infinite Sides, § whoſe Diameter may conſequently be taken for the Axis of the Sphere. Hence its Surface is, (*by the preceeding Theorem,*) equal to the Rectangle made of its Axis by the Circumference of the Circle or Polygon, by the Revolution of which it was formed, the Diameter whereof is conſequently the ſame as that of the Sphere; *Q. E. D.*

Theorem

* *B.* 3. *n.* 56. † *Sup. n.* 82. ‡ *Sup. n,* 75. § *B.* 2. *n.* 133. ‖ *B.* 4. *n.* 27.

* *B.* 4. *n.* 42. † *B.* 3. *n.* 56. ‡ *B.* 3. *n.* 17. § *Sup. n.* 66.

87 THEOR. 11. THE *Surface of a Sphere is equal to the Superficies of its circumſcribing Cylinder, which hath conſequently the ſame Height or Axis.*

THE Surface of the Sphere AMNC is equal to a Rectangle contained under its Axis, and the Circumference of a Circle having the ſame diameter MN.[*] Now the Surface of the circumſcribing Cylinder, (*whoſe Sides DP and EQ are equal to AC the Axis of the Sphere*) is equal to the ſame rectangle; for it is equal to the rectangle made of PD by the circumference of the Circle of its Baſe having for its diameter PQ equal to MN; ſeeing that the Diameter of a Sphere inſcribed in a Cylinder ought to be equal to that of the Cylinder's baſe, according to the Idea which has been given of inſcribed Figures.[†]

88 THEOR. 12. IF *a Sphere inſcribed in a Cylinder be divided by Planes perpendicular to its Axis, the Surface of each Part of the Sphere is equal to that of its correſponding Part of the Cylinder.*

AC the Axis of the Sphere is the Height of the circumſcribing Cylinder, therefore the Cylinder touches the Sphere by its two Baſes. Divide the Axis AC by perpendicular Planes, which will alſo divide the Cylinder. I ſay that the Surface of the part MHIN is equal to that of the Part MGFN of the Cylinder, as is alſo the ſurface of HAI to that of EFGD. For this Sphere may be conſidered as a Spheroid,[‡] and the Parts MHIN and HAI as portions of Spheroids. Therefore the Surface of MHIN is equal to the Rectangle of BO by the Circumference of a Circle whoſe Diameter is MN[§] which Rectangle is equal to the Surface FGMN;[‖] ſo likewiſe the Surface of HAI is equal to the Rectangle contained under AB and the Circumference of a Circle whoſe Diameter is GF,[*] which the Surface of the Part DEFG is equal to.[†]

89 PROBLEM 1. TO *divide a Sphere by a Plane, ſo that the Surfaces of the Portions of that Section may be in a given Ratio.* Archimedes 2. Prop. 4 and 5.

THE Sphere muſt be inſcribed in a Cylinder, then the Sides of the Cylinder being divided according to the given Ratio, and Lines or Parallel Planes drawn thro' the Points of that Section they will vide the Sphere according to the given Ratio; for the Surfaces of the Portions of the Sphere contain'd between the Parallels, are equal to thoſe of their correſponding Cylindrical ones, *as has been proved.*

90 THEOR. 13. THE *Surface of a Sphere is equal to that of four times its greateſt Circle.* Archimedes 1. Prop. 37.

LET us conceive a Sphere as inſcribed in a Cylinder, whoſe Baſe is conſequently equal to the greateſt Circle of the Sphere, and its Height is the Diameter thereof; conſequently the Cylindrical Superficies is equal to four times the Surface of the Circle.

---

[*] *Sup. n.* 86. [†] *Sup. n.* 46. [‡] *Sup. n.* 66.
[§] *Sup. n.* 84. [‖] *Sup. n.* 68. [*] *Sup. n.* 84
[†] *Sup. n.* 68.

‡ Now the Surface of the Sphere is equal to this Superficies: § therefore it is equal to four times its greateſt Circle.

91 THEOR. 14. THE *Surface of a Sphere is equal to that of a Circle, whoſe Radius is equal to the Diameter of the greateſt Circle of the Sphere.*

LET X be a Sphere and Z a Circle, whoſe Radius is equal to the Diameter of the greateſt Circle of the Sphere X. It muſt be proved that the Surface of X is equal to that of the Circle Z. Having ſuppoſed the Diameter of the greateſt Circle to be 1; then, *according to the Hypotheſis,* the Diameter of Z is 2. Now the Surfaces of Circles being one to another as the Squares of their Diameters, ‖ ſeeing that the ſquare of 1 is 1, and that that of 2 is 4, *according to the Hypotheſis,* the Surface of Z is quadruple that of the Circle of the Sphere X. Now the Surface of that Sphere is quadruple that of its greateſt Circle by the preceeding Theorem, it is therefore equal to that of Z; *which was to be proved.*

92 THEOR. 15. THE *whole Surface of a Cylinder,* that is, *all its circumference from its two Baſes, is to that of its inſcribed Sphere, in a* Seſquialter Ratio. Archimedes 1. Prop. 39.

1. THE Circumference of the Cylinder only is equal to the Surface of the Sphere. * Each of the Cylinder's Baſes is the greateſt Circle of the Sphere, which is the fourth Part of its Surface; † hence the two Baſes of the Cylinder are the half of the Sphere's Surface. Wherefore the whole Surface of the Cylinder is equal, 1. to once the whole Surface of the Sphere. 2, To half that Surface; therefore the Ratio is *Seſquialter, that is,* as 3 to 2.

93 THEOR. 16. A *Sphere being divided by a Plane, into two Parts, the Surfaces thereof are one to another as the Surfaces of thoſe Circles whoſe Radii are the Chords of half thoſe Portions, which ſaid Portions are equal to thoſe Circles.*

LET X be a Sphere divided by the plane BC; it muſt be proved that the Surfaces of thoſe two Portions are one to another as the Surfaces of thoſe Circles, whoſe Radii are AB and BD the Chords of half thoſe two Portions, and that they are equal thereto. 1. Having inſcribed the Sphere X in a Cylinder having the ſame Height, the Surface of the Portion ABC is equal to the circumference of its correſponding Part of the Cylinder, as is that of BCD to the other Part of the Cylinder, ‡ the Circumferences of theſe two parts are one to another as AE to ED. § Now the Squares of AB and BD are alſo as AE to ED: ¶ Therefore the Surfaces of the Portions of this Sphere are one to another as thoſe two Squares, or as the Circles whoſe Radii they are. ‖ The Square of AD is equal to the Squares of AB and BD; * therefore the Surface of the Circle whoſe Radius is AD, is equal to the Surfaces of the two Circles, whoſe Radii are AB and BD. † Now the Surface of the Circle whoſe Radius is AD, is equal to that of the Sphere X: ‡ therefore this Surface is equal to that of the two Circles whoſe Radii are AB and BD. But it was ſaid that they are alſo one to another as theſe Portions; therefore they are each equal to their correſponding one, ſeeing the ſame whole is divided in the ſame Proportion.

---

‡ *Sup. n.* 72. § *Sup. n.* 87. ‖ *B.* 4, *n.* 93.

* *Sup. n.* 87. † *Sup. n.* 90,

‡ *Sup. n.* 88. § *Sup. n.* 69. ¶ *B.* 4. *n.* 80.

‖ *B.* 4. *n.* 93.

THEOR. 17. THE *Surface of a Sphere is double that of the Circumferance of the inſcribed Cylinder, having its Height equal to the Diameter of its Baſe.* 94

THE Surface of the Sphere Z is Quadruple that of the Circle, whoſe Diameter is A. * The Surface of the Circle whoſe Diameter is A, is equal to the Surface of the Circles whoſe Diameters are C and B, ſeeing that theſe Surfaces are as the Squares of their Diameters, and that $AA = CC + BB$. † Now $C = B$: Therefore the Surface of the Circle whoſe Diameter is A, is equal to twice that of the Circle whoſe Diameter is C; and conſequently ſince that the Surface of A is the fourth Part of the Surface of the Sphere Z, that of the Circle whoſe Diameter is C is the eighth Part of that of the Sphere Z. Now the Surface of the ſame Circle, whoſe Diameter is C, is the fourth Part of the inſcribed Cylinder's Surface: ‡ Therefore this Circumferenee is half the Surface of the Sphere Z.

SECT. IV.

OF the Solidity of Solids.

EVIDENT Propoſitions concerning their Solidity.

PROPOSITION I.

EVERY *Parallelopipedon may be conceived as deſcribed by its Plane or Baſe, moving always parallel to itſelf.* 95

PROP. 2. IF *the Direction of the Plane whoſe motion deſcribes the Parallelopipedon, be according to its perpendicular Height, it is right and its Angles are right. If it is according to an Oblique Line, it is Oblique, and then its Angles are not right ones.* 96

PROP.

---

* *B.* 4, *n.* 78. † *B.* 4. *n.* 93. ‡ *Sup n.* 91.

* *Sup. n.* 70. † *B.* 4. *n.* 78. ‡ *Sup. n.* 72.

97 **Prop.** 3. A *Parallelopipedon may alſo be conceived as compoſed of innumerable Planes all equal, and parallel to that which is its Baſe.*

98 **Corollary.** It *is the ſame of Cylinders, and Priſms.*

99 **Prop.** 4. A *Pyramid may be conceived as made by its Baſe, moving always Parallel to itſelf, and decreaſing in proportion as it riſes. If the Motion be according to its perpendicular Height, the Pyramid is a right Angled one; if it move Obliquely, it is an Oblique one.*

100 **Prop.** 5. It *may be conceived that a Pyramid is compoſed of innumerable parallel Planes, which diminiſh in Proportion to their Height.*

101 **Prop.** 6. Two *Solids having the ſame Height contain an equal number of Planes,* that is, *there may be conceived as many in one as in the other.*

102 **Prop.** 7. If *an Oblique Parallelopipedon, or Oblique Priſm, every Pyramid, every Cone, are conceived as compoſed of innumerable Planes, their Surfaces are even.*

Consider the Pyramid BAC. If the Planes O, O, O, O, O, O, which compoſe it are thick, it is evident that its Surface is not even, but by Degrees ſo much greater, as the Planes O are thicker; and they are ſo much leſſer, as there are more of them. If it is then ſuppoſed that they are infinite in Number, they would have no thickneſs; and then the Surface of BAC is without Degrees. Which is to be conceived of every Parallelopipedon, Priſm, Pyramid and Cone.

103 **Prop.** 8. A *Parallelopipedon may be conceived as deſcribed upon a given Line, Similar, and in the ſame manner poſited as another given Parallelopipedon.* Eucl. 11. Prop. 27.

Euclid propoſes the doing this Thing, (*which is eaſy,*) but it is ſufficient to conceive it as being done.

104 **Prob.** 9. Two *Solids are equal, which are compoſed of an equal Number of Planes equally thick, ſimilar and equal.*

If for Example the two Solids X and Z were each compoſed of a Million of Planes equal and of an equal thickneſs, they are evidently equal. I add the Word *Similar,* to ſhew that in caſe the Planes which compoſe X aſcend and diminiſh as thoſe of Z ariſing in like manner diminiſh, each being Similar to its correſponding one, *that is,* the hundredth Part of X being equal and ſimilar to the hundredth Part of Z, it muſt neceſſarily be that X and Z are equal.

105 **Prop.** 10. If *the Parallelopipedon X be divided according to its Diagonal AC or EG, it is divided into two equal Triangular Priſms.* Eucl. 11. Prop. 28. See P. B. Plate 9. Fig. 36. and 37.

The two Parts of X have equal Baſes, and the ſame Height: Therefore by the preceeding Propoſition, they are equal. According to the Definition of Priſms, * they are Triangular ones.

106 PROP. 11. IF *the Sides of the oppoſite Planes of a Parallelopipedon are divided into two equal Parts, and that Planes be drawn thro' the Sections; the Line of common Section of the Planes, and the Diameter of the Parallelopipedon, are divided equally in two* Eucl. 11. Prop. 39.

THIS is evident, for the Diameter is the Baſe of a Triangle cut Parallel to one of its Sides thro' the Middle of the other; which divides the Side and the Baſe equally.

107 PROP. 12. IF *a Solid is bounded by parallel Planes, the oppoſite Planes are ſimilar and equal Parallelograms.* Eucl. 11. Prop. 24. (*ſame figure*)

1. THE Lines AD and BC are Parallel, † as alſo AB and DC. It is the ſame of the Lines EF and HG, and of FG and EH. Therefore the Planes ABCD and EFGH are Parallelograms. 2. Theſe Parallelograms are Similar; for the Angle EFG is equal to ADC: ‡ &c. 3. The Parallels AB and HE between the Parallels AE and BH are equal, whether they be Oblique or Perpendicular. § Therefore it is demonſtrable that the oppoſite Planes are equal.

108 THEOR. 1. EVERY *Section of a Parallelopipedon, Priſm, Cylinder, Pyramid, Cone, which is made Parallel to its Baſe, is Similar to its Baſe.* Eucl. 11. Prop. 25.

1. IN the Parallelopipedon, Priſm, and Cylinder, which are made by their Baſes moving always Parallel, this is evident. 2. Alſo in the Pyramid, and Cone, which are Solids made by the Motion of their Baſe, which dimiſhes Proportionably, therefore the parallel Planes of which either of them may be conceived to be made, are all Similar. Now the Sections are ſome one of theſe Planes; conſequently they are Similar to the Baſe; *which was to be proved.* For a farther Proof hereof, ſee P. B. Plate 9. Fig. 35. Which ſhews a real Parallelopipedon, cut Parallel to its Baſe.

109 THEOR. 2. ALL *Solids of the ſame Name, (having the ſame Height, and equal Baſes,) are equal, whether they be right, or Oblique.* Eucl. 11. Prop. 29. 30. & 31.

THAT *is*, that Solid Parallelopipedons being upon equal Baſes, and of the ſame Height, are equal one to another, they have an equal Number of Planes all ſimilar and equal to their Baſes; ‖ therefore they ought to be equal. It is the ſame of Priſms, Cylinders, Pyramids, and Cones; for each Plane of one is equal and ſimilar to each of the other taken at the ſame Height.

110 COROLLARY. THEREFORE *in meaſuring Solids regard is had only to their Height, and Baſe.*

---

* *Sup. n.* 34. † *Sup. n.* 34. ‡ *Sup. n.* 36.
§ *B.* 2. *n.* 110. ‖ *Sup. n.* 101.

For an Oblique Parallelopipedon hath a greater Surface than a right one. Notwithſtanding if it be upon an equal Baſe, and of the ſame Height, it is not greater in its ſolid Content.

111 Theor. 3. Solid *Parallelopipedons and Priſms having the ſame Height, are one to another as their Baſes; and if they have the ſame Baſe, they are as their Heights.* Eucl. 11. Prop. 32.

Let the two Solids be Z and Y. 1. They have an equal number of Planes, Parallel and Similar to their Baſes. If their Baſes are equal, then they are equal; if that of Z is double that of X, then all the Planes of Z being double thoſe of X, Z will be the double of X. 2. If the two Solids have equal Baſes, they are as the Heights, for all their Planes being equal, if the Solid Z is twice as high as X, it ought to have twice as many Planes. If it is three times higher, it will have three times as many.

112 Theor. 4. Two *Cylinders of the ſame Height are as their Baſes, or if they have one and the ſame Baſe, they are as their Heights.* Eucl. 12. Prop. 11 and 14.

This has been demonſtrated of the Priſm, and conſequently of the Cylinder, which may be taken for a Priſm.

113 Theor. 5. If *a Cylinder is divided by a Plane parallel to the Planes oppoſite to its Baſes, the Segments of the Cylinder are one to another as the Segments of its Axis.* Eucl. 12. Prop. 13.

The Planes of the Sections are equal. * therefore all the Segments are as ſo many Cylinders whoſe Baſes are equal, and therefore, by the preceeding Theorem, they are one to another as their Heights, which are the Segments of the Axis.

114 Theor. 6. A *Parallelopipedon is double a Triangular Priſm of the ſame Height, if the Baſe of the Parallelogram is double that of the triangular Priſm.* Eucl. 11. Prop. 40.

This is evident, ſeeing that the two Priſms are as their Baſes. † See P. B. Plate 9. Fig. 38 and 39.

115 Theor. 7. Every *Polygonal Priſm may be divided into triangular Priſms.*

Let K be a Polygonal Priſm whoſe Baſes are ABCDE and GHILF: Reduce the Polygonal Baſes into Triangles. By the Definition of triangular Priſms, the Solids ABCGHI, ACDGIL, ADEGLE are triangular Priſms: therefore the Priſm K may be divided into triangular Priſms.

116 Theor. 8. A *Priſm is equal to ſeveral Priſms having the ſame Height, if its Baſe is equal to thoſe of all the Priſms. It is the ſame of Pyramids.*

For let there be conceived in theſe Solids Planes parallel to the Baſe, ‡ there will be an equal number of Planes in each; § the Planes in the greateſt Priſm are each equal to all the Planes which are in the other Priſms; for it is to them as its Baſe is to all the Baſes of thoſe Priſms. Now it is equal thereto, therefore, &c. It is the ſame of Pyramids.

---

* *Sup. n.* 98. † *Sup. n.* 111. ‡ *Sup. n.* 97.
§ *Sup. n.* 101.

COROLLARY 1. *A Cylinder is equal to a triangular Priſm of the ſame Height, whoſe Baſe is equal to that of the Cylinder.* 117

A Cylinder is a Polygonal Priſm. Every polygonal Priſm may be divided into triangular Priſms. * theſe Priſms are equal to one only triangular Priſm of the ſame Height, whoſe Baſe is equal to all thoſe of the Priſms. † Therefore the Cylinder equal to the polygonal Priſm, is ſo likewiſe to the triangular Priſm which hath the ſame Height, and whoſe Baſe is equal thereto.

COROL. 2. THEREFORE *a Cylinder X is equal to ſeveral Cylinders A, B, C, &c. of the ſame Height, whoſe Baſes all taken together are equal thereto.* 118

FOR all theſe Cylinders are equal to ſo many triangular Priſms, having the ſame Height and equal Baſes. That to which X is equal, (*and therefore of the ſame Height and upon an equal Baſe,*) is equal to all the other triangular Priſms, and therefore to the Cylinders A, B, C, &c. whoſe Baſes all together are equal to that of X; wherefore X is equal to A, B, C, &c.

THEOR. 9. *A triangular Priſm may be divided into three equal triangular Pyramids.* Eucl. 12. Prop. 7. 119

LET X be a triangular Priſm; upon each of its three Sides draw Diagonals, which make ſix Triangles. The Triangles BAD, BCD are equal, ‡ the Pyramids BADF and BDCF which have the ſame Vertex, and therefore the ſame Height are equal. § theſe two Pyramids, being conceived as deducted from the Priſm X, there remains a third *viz.* FCED, which hath in the firſt Place the ſame Vertex D, and therefore the ſame Height as the Pyramid FBCD; they have equal Baſes, *viz.* the equal Triangles EBC and FCE: ‖ Therefore they are equal. * Now the Pyramid FBCD is the ſame as BDCF, being formed by the ſame Triangles: therefore the Pyramids FCED and BADF are equal, being ſo to a Third; therefore the Priſm X may be divided into three equal Pyramids, which are BADF, BDCF, and CEDF. See alſo P. B. Plate 10. Fig. 41.

COROL. 1. THEREFORE *every Pyramid is a third Part of the whole Priſm of the ſame Height, which is upon the ſame, or an equal Baſe.* 120

COROL. 2. THEREFORE *to meaſure a Pyramid, the Baſe muſt be multiplied by one Third of its height.* 121

LEMMA. *A Polygonal Pyramid may be divided into triangular Pyramids.* 122

THE Baſe of a polygonal Pyramid is a Polygon, conſequently it may be reduced into Triangles, upon which Planes being conceived to as raiſed along the Sides of that Pyramid even to its

---

* *Sup. n.* 115. † *Sup. n.* 116. ‡ *B.* 2. *n.* 128.

§ *Sup. n.* 109. ‖ *B.* 2. *n.* 128.

* *Sup. n.* 109.

its Vertex, there will be ſeveral triangular Pyramids, which are the Parts of the Polygonal Pyramid.

123 THEOR. 10. PYRAMIDS *having triangular Baſes, and which are of the ſame Height, are one to another as their Baſes. It is the ſame of thoſe which have polygonal Baſes.* Eucl. 12. Prop. 5 and 6.

PRISMS are triple the Pyramids of the ſame Height and Baſe. * But Priſms of the ſame Height are one to another as their Baſes, or thoſe of the ſame Baſe are as their Heights: † Therefore Pyramids which are only the one Third thereof, are ſo likewiſe, by the preceeding Lemma, ſeeing that polygonal Pyramids may be reduced into Triangular ones, they may be taken for each other promiſcuouſly.

124 THEOR. 11. EVERY *Pyramid having a Triangular Baſe may be divided into two equal Pyramids, ſimilar one to another, and to the whole Pyramid, and into two equal Priſms, which two Priſms are together greater than the half of the whole Pyramid.* Eucl. 12. Prop. 3.

125 THEOR: 12: TWO *Pyramids of the ſame Height having triangular Baſes, may be divided into two other Pyramids equal one to another and Similar to the whole Pyramid, and into two equal Priſms; and if the Pyramids proceeding from this Diviſion, are continually divided in the ſame Manner: As the Baſe of one of the Pyramids is to the Baſe of the other; ſo all the Priſms which are in one of the Pyramids, are to all the Priſms of the other being equal in Number.* Eucl. 12. Prop. 4.

THE Demonſtrations of theſe two Propoſitions are here omitted, as being difficult to conceive by the bare Repreſentation thereof upon a Plane; I therefore refer the Reader to P. B. Plate 10. Fig. 40. Which contains a Pyramid exactly divided, according to the preceeding Theorem, from whence the other may be eaſily comprehended.

126 THEOR. 13. EVERY *polygonal Pyramid is the third Part of the whole Priſm of the ſame Height, and which is upon the ſame or an equal Baſe.*

FOR having reduced the Baſe of each of theſe two Solids into Triangles, the polygonal Pyramid will be divided into triangular Pyramids. Now each of theſe triangular Pyramids, is the third Part of each of the triangular Priſms: ‡ the whole polygonal Pyramid is therefore the third Part of the whole polygonal Priſm.

127 PROBLEM. TO *find the Solidity of Z the Fruſtum of a Pyramid, whoſe Baſes are Parallel; let that of the Bottom be for Example 36, the top One 9, the Number 18 is a mean Proportional between theſe two Numbers.*

RULE. Theſe two Numbers added into one Sum, and that Sum multiplied by 2, which I ſuppoſe to be the one Third of the Fruſtum's Height, the Product 126 is the Fruſtum's Solidity.

---

* *Sup. n.* 120. † *Sup. n.* 111. ‡ *Sup. n.* 120.

To prove the Truth hereof; let the bottom Baſe be a, the top One b, and c the Height of the Fruſtum Z, let the Part of the Axis which is wanting to compleat the Pyramid be d, the Solidity of a Priſm which is triple the whole Pyramid, is $aac + aad$, from which I take bbd the Solidity of a Priſm, whoſe third Part is the Pyramid X: Hence $aac + aad - bbd = 3Z$. Seeing that $\div$ $aa.\ ab.\ bb$; then, according to the Rule, theſe three Terms which are the ſame as 36, 18, 9, being multiplied by the Third of c, will give the Solidity of Z; ſo multiplying them by the whole c, I ſhall have $aac + abc + bbc = 3Z$: it remains to be demonſtrated that $aac + aad - bbd = aac + abc + bbc$: Take away aac from each part, there remains $aad - bbd = abc + bbc$. it muſt be ſhewn that this is ſo. $a - b.\ b :: c.\ d$ multiplying $a - b$, and b by $a + b$, * $aa - bb.\ ab + bb :: a - b.\ b$. therefore $aa - bb.\ ab + bb :: c,\ d$: † then multiplying the Extremes and the Means, there is $aad - bbd = abc + bbc$; ‡ which remained to be proved. The Rule is therefore true. The true Form of the Fruſtum of a Pyramid, is exhibited in P. B. Plate 7. Fig. 26.

128 THEOR. 14. A *Cone is the Third of a Cylinder of the ſame Height their Baſes being equal.* Eucl. 12. Prop. 10.

A Cone is a Pyramid of infinite Sides: Now a Pyramid is the third Part of a Priſm of the ſame Height, having an equal Baſe: § therefore the Cone is alſo the third Part of a Cylinder of the ſame Height, and upon the ſame or an equal Baſe, ſeeing that a Cylinder is a Priſm having an infinite number of Sides.

129 COROLLARY 1. A *Cone is equal to all thoſe Cones which have the ſame Height, and whoſe Baſes all taken together are equal to the Baſe thereof.*

CONES are Pyramids of an infinite number of Sides, which conſtitute the third Part of the Cylinder or Priſm; therefore this Propoſition is the ſame as that which has been propoſed, ‖

130 COROL. 2. TWO *Cones of the ſame Height are as their Baſes; and if they have the ſame Baſe, they are as their Heights, which is evident.* Euclid 12. Prop. 11 and 14.

131 THEOR. 15. SIMILAR *Priſms and Parallelepipedons are in a Ratio compounded of the Ratio's of their three Dimenſions, and that Ratio is triple.* Eucl. 11. Prop. 33.

THEIR Solidity depends upon the Multiplication of their Dimenſions. * The Ratio which they have one to another is compounded of that of their three Dimenſions. † Which being ſimilar, the Ratio is triple. ‡

132 THEOR. 16. SIMILAR *Cylinders are one to another in Ratio compounded of their Dimenſions, and that Ratio is triple.* Eucl. 12. Prop. 12.

---

* *B.* 3. *n.* 54. † *B.* 3. *n.* 53. ‡ *B.* 3. *n.* 56.
§ *Sup n.* 120. ‖ *Sup. n.* 116.
* *Sup. n.* 110. † *B.* 3. *n.* 78. ‡ *B.* 3. *n.* 74.

For Cylinders being as Priſms having Baſes of an infinite Number of Sides, they are one to another as theſe Priſms, by the preceeding Theorem.

133 Theorem 17. Similar *Pyramids are in Ratio compounded of the Ratio's of their three Dimenſions, and that Ratio is triple.* Eucl. 12. Prop. 8.

This is evident, for they are the third Part of the Priſms which are upon their Baſes, and which have the ſame Height, * and therefore in the ſame Ratio.

134 Theor. 18. Similar *Cones are one to another in Ratio compounded of the Ratio's of their Dimenſions, and that Ratio is triple.* Eucl. 12. Prop. 12.

This is a Conſequence which naturally follows, from the Similarity which is between the Cone and a Pyramid, whoſe Baſe has an infinite Number of Sides.

135 Theor. 19. Similar *Cylinders are one to another as the Cubes of the Diameters of their Baſes.*

They are in Ratio triple that of each of their Dimenſions, and conſequently of the Ratio of the Diameter of their Baſes. Now the Cubes of their Diameters are in Ratio triple that of the ſame Diameters. † therefore ſeeing that Ratio's compounded of equal Ratio's are equal, ſimilar Cylinders are one to another as the Cubes of the Diameters of their Baſes. It is the ſame of ſimilar Cones, which is demonſtrable in the ſame manner.

136 Theor. 20. Equal *Parallelopipedons have their Heights and Baſes reciprocal; and if they are reciprocal, the two Solids are equal.* Eucl. 11. Prop. 34.

Let X and Z be two equal Parallelopipedons, A the Height of X, and B that of Z, let the Content of the Baſe of X be M, and N that of the Baſe of Z. According to what is ſuppoſed, $A \times M = B \times N$: then $A . B :: N . M$; ‡ wherefore theſe four Quantities A, B, N, M are reciprocal. § Now if theſe four Quantities are reciprocal, *that is*, If A, M, B, N may be ſo ranged that $A . B :: N . M$, then will $A \times M = B \times N$. ‖

137 Theor. 21. Two *Cylinders being equal,* (It is the ſame of Pyramids and Cones,) *their Heights and Baſes are reciprocal; and if they are reciprocal, the two Cylinders are equal.* Eucl. 12. Prop. 9 and 15.

The Demonſtration of the preceeding Theorem will ſerve for this; by only ſuppoſing X and Z to be two Cylinders.

138 Theor. 22. If *three right Lines A, B, C, are proportional, the Solid Parallelopipedon ABC compoſed of theſe three Lines, is equal to the Parallelopipedon BBB made of the mean Term B, provided that theſe Solids are Equiangular.* Eucl. 11. Prop. 36.

∺ A. B.

---

* *Sup. n.* 120. † *B.* 3. *n.* 87. ‡ *B.* 3. *n.* 58.
§ *B. n.* 3. 61. ‖ *B.* 3. *n.* 56.

∺A. B. C. then AC=BB.‡ then AC and BB multiplied by B, the Product ACB is alſo equal to the Product BBB.§ Now ACB is the Parallelopipedon made of the three Lines A, B, C, equal therefore to BBB which is made of the mean Line B.

Theo. 23. *A, B, C, D, are four proportional Lines. The ſimi-* 139
lar Solids conſtructed upon theſe Lines are proportional; and if they are in Proportion, the four Lines are proportional. Eucl. 11. Prop. 37.

This has been proved.¶

Theo. 24. *A Sphere is equal to a Cone, or Polygonal Pyramid,* 140
whoſe Axis is the Radius of that Sphere, and its Baſe a Circle, whoſe Radius is the Diameter of that ſame Sphere.

1. By conceiving infinite Cones or polygonal Pyramids, whoſe Vertices are in the Center of the Sphere, and the Baſes in the Surface of the ſame Sphere; it is evident that it may be ſaid, that the Solidity of that Sphere is equal to all thoſe Cones or polygonal Pyramids, ſeeing 'tis allowed, that *the whole is equal to all its Parts taken together.* 2. All theſe Cones are equal to a Cone having the ſame Height, (*viz. the radius of that Sphere,*) and for its Baſe the Surface of that Sphere, (*which is equal to the Baſes of theſe Cones.*)* Now the Surface of that Sphere is equal to that of a Circle, whoſe radius is the diameter of that Sphere: † the Solidity is therefore equal to that of a Cone whoſe
Baſe, &c. Suppoſe the ratio of the diameter of a Circle to its 141
Circumference be as 7 to 22, it may be ſaid that the Solidity of the Sphere is to the Cube of its diameter as 11 to 21: for let *m* be the Circumference of the Circle, and *n* the diameter. 1. *mnn. nnn. m. n.*‡ Now *m* is to *n* as 22 to 7, or 66 to 21. therefore the ſixth Part of *mnn*, which is the Solidity of the Sphere, (as is eaſily perceived from what has been heretofore demonſtrated,) is to *nnn* the Cube of its Diameter, as the ſixth Part of 66, that is, as 11 is to 21; *which was to be proved.*

Theorem 25. The *ratio of the Cylinder X to its incribed Sphere* 142
Z, is ſeſquialter.

Let B and C be two Cones whoſe Axis is the radius of the Sphere Z, and that the radius of the Baſe of B be that of the Sphere Z, and the radius of the Baſe of C be the Axis or diameter of the Sphere X, then theſe two Cones B and C are one to another as their Baſes.§ Now that of C is quadruple that of B. therefore the Cone C is quadruple the Cone B; hence B. C∷ 1. 4. the leſſer Cone B is the ſixth Part of the Cylinder X, whoſe Baſe is the greateſt Circle of the Sphere Z, and its Axis the diameter thereof; for the Cone B is the third Part of a Cylinder, having the ſame Baſe and Axis; ‖ conſequently it is the ſixth Part of a Cylinder having twice that Axis, and the ſame Baſe; hence X. B∷ 6. 1. The Cone C is equal to the

---

‡ *B.* 3. *n.* 37. § *B.* 3. *n.* 54. ¶ *B.* 3. *n.* 85.

* *Sup. n.* 129. † *Sup. n.* 91, ‡ *B.* 3. *n.* 54,

§ *Sup. n.* 130. ‖ *Sup. n.* 128.

Sphere Z,¶ it has been proved that B. C:: 1. 4; hence B. Z:: 1. 4: therefore ſeeing that the Cylinder X contains ſix ſuch Parts, as the Sphere X contains four, X. Z:: 6. 4; which is a ſeſquialter Ratio.

143 THEOREM 26. SPHERES *are one to another as the Cubes of their Diameters, or in Ratio triple that of their Diameters.* Eucl. 12. Prop. 18.

SPHERES are in a Ratio compounded of the Ratio's of their three Dimenſions; all the Parts of Spheres are ſimilar; therefore there three Dimenſions have the ſame Ratio; therefore the Ratio which they compoſe is triple that of each of the Ratio's of their Dimenſions; for example, of that of their Diameters. Now the Cubes of the Diameters are in triple Ratio of that of theſe Diameters: Spheres are then one to another as the Cubes of their Diameters.

SECT. V.

OF the Method of inſcribing or circumſcribing a Sphere with the five regular Bodies.

ADVERTISEMENT.

144 THE true Forms of thoſe Bodies are ſhewn in P. B. Plate I. Fig. 1, 3, 4, 5, 6.

145 THEOREM 1. EVERY *Section of a Sphere by a Plane, is a Circle.*

X is the Section of a Sphere whoſe Center is A; it muſt be proved that this Section is a Circle: to do, which let us conceive, 1. That a Perpendicular AB is let fall from A the Center of the Sphere upon the Plane of this Section, which I call X. 2. That Lines, ſuch as AC, be drawn from the ſame Center A to all the extreme Parts of X: theſe Lines (*which are all Radii of the Sphere,*) are equal, they are Oblique, by reaſon there can be drawn but one Perpendicular from A to X. Now oblique Lines which are equal have their Bottoms equally diſtant from the Perpendicular,* therefore all the Lines drawn from the Extremities of X to the Point B are equal, and the Extremities are conſequently in the Circumference of a Cirele; therefore X is a Circle, according to the Definition thereof.†

146 PROBLEM 1. TO *inſcribe a regular Polygon having even an Number of Sides, in the greateſt of two unequal concentric Circles, ſo that the Polygon may not touch the leſſer Circle.* Eucl. 12. Prop. 16.

PROP.

¶ *Sup. n.* 140. * *B.* 1. *n.* 61. † *B.* 1. *n.* 20.

PROB. 2. TO *inſcribe in the greateſt of two unequal Spheres having one and the ſame Center, a Polyedron whoſe Planes touch not the Surface of the leſſer Sphere.* Eucl. 12. Prop. 17. 147

THESE two Problems appear to me as unneceſſary, therefore I paſs them.

LEMMA. 1. IF aa, *the Square of AC, is triple of* bb, *the Square of CD a mean proportional between AD and BD, I ſay that DB is the third Part of the diameter AB.* 148

LET AD be called *c*, 1. $aa = bb + cc$,* and becauſe that by the Suppoſition $3bb = aa$: then $3bb = bb + cc$: taking away *bb* from each ſide, it will be $2bb = cc$. 2. ∺ AD. CD. BD. or ∺ *c. b. BD* by the Hypotheſis: then *cc* or *2bb. bb* ∺ AD. DB.† therefore AD is double BD, and conſequently DB is the third of AB; *which was to be proved.*

THEOR. 2. THE *Square of one of the Sides of a* Tetraedron, *or Equilateral Pyramid, is equal to ſix Times the Square of the third Part of the diameter of its circumſcribing Sphere.* 149

THE *Tetraedron* or Equilateral Pyramid, is compoſed of four equal and Equilateral Triangles. Let us imagine a *Tetraedron* to be inſcribed in the Sphere X, one of whoſe Sides is AC, or *a*, and that CD is the Radius of the Circle in which one of the Equilateral Triangles which compoſe the Solid is inſcribed: $\overline{AC}^2 = 3\,\overline{CD}^2$, or $aa = 3bb$;‡ conſequently DB is the third Part of AB the Diameter of the Sphere X, by the preceeding Lemma. Then let $BD = c$, and conſequently $AB = 3c$. Since that ∺ *3c. a. 2c*:§ then $6cc = aa$;‖ Q. E. D.

COROLL. THE *Square of the diameter of a Sphere, is in ſeſquialter ratio to the Square of one of the Sides of its inſcribed* Tetraedron. Eucl. 13. Prop. 13. 150

IT has been proved that *aa*, the Square of the *Tetraedron's* Side, is equal to ſix Times the Square of *c*, the third Part of the diameter of the Sphere. Now the Square of *3c*, the diameter of the Sphere is *9cc*; therefore the ratio of the ſide of the *Tetraedron* to that of the diameter of the Sphere is as *6cc* to *9cc*, or 6 to 9, which is a ſeſquialteral Ratio.

THEOR. 3. THE *ſide of the* Tetraedron *is incommenſurable in itſelf, and commenſurable in Power, with the diameter of its circumſcribing Sphere.* 151

LET AC or *a*, be the ſide of the *Tetraedron* as above, and AB or *3c* the diameter of the Sphere, by what has been demonſtrated in the preceeding Theorem, DB is equal to *c*, and $AD = 2c$; hence ∺ *3c. a. 2c*:¶ therefore *9cc. aa*:: *3c. 2c.** Now 3 and 2 are not ſquare Numbers; therefore *a* is incommenſurable in itſelf with *3c*, and commenſurable in Power.† In the preceeding

---

* *B.* 4. *n.* 78.

† *B.* 3. *n.* 86.

‡ *B.* 4. *n.* 146.

§ *B.* 4. *n.* 28.

‖ *B.* 3. *n.* 57.

¶ *B.* 4. *n.* 28.

* *B.* 3. *n.* 86.

† *B.* 4. *n.* 115.

ing Theorem, we proved that *aa* is to the Square of the Sphere's Diameter as 6 to 9.

152 PROB. 3. TO *inſcribe a* Tetraedron *in a Sphere, or to find a Circle ſufficient to contain one of the Sides of the* Tetraedron. Eucl. 13. Prop. 13.

DIVIDE AB the diameter of the Sphere into three equal Parts, ſo that DB be double AD, and upon D erect the Perpendicular DC, which is the Radius of a Circle, in which an Equilateral Triangle being made whoſe ſide is BC, you will have one of the Sides of a *Tetraedron*, as is evident.*

153 THEOR. 4. THE *Square of the diameter of a Sphere is triple the Square of each ſide of its inſcribed* Cube, *or* Hexaedron. Eucl. 13. Prop. 15.

THE *Cube* or *Hexaedron* X is inſcribed in a Sphere. Let the diagonal $AB = m$, which is the diameter of the Sphere, let the diagonal of one of the Sides of the *Cube* be BD. I call s all the Sides of the *Cube*, which are all equal. $\overline{AB}^2 = \overline{BD}^2 + \overline{AD}^2$, or $mm = nn + ss$.† ſo likewiſe $\overline{BD}^2 = \overline{BC}^2 + \overline{CD}^2$, or $nn + 2ss$. then ſubſtituting in the room of nn its equal $2ss$, it will be $mm = 3ss$. Q. E. D.

154 PROB. 4. THE *diameter of a Sphere being given, to find the ſide of the* Cube *or* Hexaedron *which may be inſcribed therein, or to find a Circle which may contain one of the Sides of the* Cube. Eucl. 13. Prop. 15.

LET AB be the diameter of a Sphere in which muſt be inſcribed a *Cube*, divide it into three ſuch Parts, that BD be double of AD: upon D raiſe the Perpendicular CD, and from C draw a Line to A, which is the ſide of the *Cube* required. For let $BA = 3c$, and $CA = d$: then $\div\, 3c.\ d.\ c.$§ then $3cc = dd$.‖ the Square of AB or $3c$ is $9cc$; therefore the Square of AB is triple that of d, which is only $3cc$. Wherefore AC is the ſide of the *Cube* required, by the preceeding Theorem. Then if you would have a Circle capable of containing one of the Sides of the *Cube*, there muſt be a Square deſcribed one of whoſe Sides is AC, and its circumſcribing Circle will be that which was required.

155 THEOR. 5. THE *ſide of a* Cube *is incommenſurable in itſelf, and commenſurable in Power, with the diameter of a Sphere.* (ſame Figure.)

AC or d, the ſide of the *Cube*, si a mean proportional between the whole diameter AB or $3c$, and its third Part c; hence ſince that $\div\, 3c.\ d.\ c$; then $3c.\ c :: 3.\ 1.$ the two Numbers 3 and 1 are not ſquare Numbers, therefore AC is in itſelf incommenſurable with AB; but commenſurable in Power, by reaſon its ſquare is the third Part of AB.¶

---

* *Sup. n.* 149.

† *B.* 4. *n.* 78.

§ *B.* 4. *n.* 28,

‖ *B.* 3. *n.* 57.

¶ *B.* 4. *n.* 131.

THEOR. 6. THE *ſquare of each ſide of an* Octoedron *is half that of the diameter of its circumſcribing Sphere.* Eucl. 13. Prop. 14. 156

AN *Octoedron* is compoſed of eight equal Equilateral Triangles, whoſe Sides are Chords of a Quadrant of a Circle or ninety Degrees. Now it is evident, that the Square of the Chord of ninety Degrees is half that of the Diameter: for two Chords of ninety Degrees make a right Angle, whoſe Baſe is the diameter of the Circle; therefore the Square of theſe two Chords is equal to that of the Diameter, which is conſequently double that of each of the two Chords.

THEOR. 7. THE *ſame Circle will contain the Square which is one of the Sides of a* Cube, *and the Triangle which is one of the Sides of an* Octoedron, *they being both inſcribed in the ſame Sphere.* Eucl. 14. Prop. 8. 157

LET the Square BCDE be the ſide of a Cube, inſcribed in a Circle, the ſide of an *Octoedron* inſcribed in a Circle is the Triangle FGH. It muſt be demonſtrated that if theſe two Solids are inſcribed in one and the ſame Sphere, theſe two Circles are equal. Let a be the diameter of the Sphere, b the ſide of the Square which is one of the Sides of the Cube, and c the ſide of the Triangle which is one of the Sides of the *Octoedron*. Let the Radius of that Circle which comprehends the ſide of the *Cube* be called y, and that of the Circle which contains the ſide of the *Octoedron* be x. It muſt be proved that $y=x$. 1. If bb is conceived to be the Square of the Cube inſcribed in the Circle whoſe Radius is y, it is evident that $2yy=bb$.* and ſince that $3bb=aa$: † then $aa=6yy$. 2. $3xx=cc$.‡ Now $2cc=aa$: || then $6xx=aa$. hence ſince $6yy=aa=6xx$, then $6yy=6xx$; therefore $y=x$; *which was to be proved.*

THEOR. 8. THE *ſide of an* Octoedron *is incommenſurable in itſelf, and commenſurable in Power with the diameter of its circumſcribing Sphere.* 158

THE Square of each ſide of the *Octoedron* is to that of the diameter of the Sphere, as 1 to 2.¶ Now 1 and 2 are not ſquare Numbers; therefore the ſide is incommenſurable in itſelf, with the diameter of the Sphere,* and commenſurable in Power, by reaſon its Square is half that of the Sphere.

PROP. 5. TO *find the ſide of an* Octoedron, *and a Circle capable of containing one of the Sides thereof.* 159

FIND by the fourth Problem a Circle that will comprehend one of the Sides of a *Cube*, that Circle by the ſeventh Theorem, is ſufficient to contain one of the Sides of the *Octoedron*, there is then nothing required to be done, but to inſcribe an Equilateral Triangle in that Circle: the ſide of that Triangle is the ſide of the *Octoedron* required.

THEOR. 9. DIAGONALS *or Chords being drawn upon the twelve Pentagons which compoſe the* Dodecaedron, *thoſe Diagonals which* 160

---

* *B*. 4. *n*. 78.
† *Sup*. *n*. 153.
‡ *B*. 4. *n*. 146.
§ *Sup*. *n*. 156.
|| *Sup*. *n*. 156.
* *B*. 4. *n*. 115.

join together form ſix Squares which are the ſix Sides of a Cube *or* Hexaedron *inſcribed in the ſame Sphere as the* Dodecaedron.

Let there be four pentagonal Sides of a *Dodecaedron*, K, X, Z, Y, (*in reading this keep in hand a* Dodecaedron) upon each of theſe Sides conceive Diagonals, one from A to B, from B to C, from C to D, and from D to A, ſo upon all the other Sides; it muſt be proved. 1. That theſe four Diagonals make a Square, *viz.* ABCD. 2. That the other Diagonals with theſe, form ſix Squares equal to ABCD, which compoſe a *Cube* inſcribed in the ſame Sphere as the *Dodecaedron*; therefore each ſide of the *Cube* is equal to each diagonal of the Pentagons. 1. All theſe Diagonals ſubtending equal Angles, are equal. 2. Let us conceive Lines to be drawn from the four Points A. B. C. D. *which are upon the Sphere in which the* Dodecaedron *is inſcribed* to the Center of the Sphere, they will make a quadrilateral Pyramid, whoſe Baſe is the plane which divides the Sphere and paſſeth thro' theſe four Points. 3. The Section of the Sphere by the Plane ABCD is a Circle.[*] Now there can be no Figure of four equal Sides inſcribed in a Circle, except the Square; for the four Angles are equal to four right ones: [†] and by reaſon that they ſtand upon equal Arcs, they are equal: therefore the Figure ABCD, which hath equal Sides, and which is inſcribed in a Circle, is a Square. 4. Every Pentagon may be reduced into three Triangles; wherefore the Surface of a *Dodecaedron*, conſiſting of twelve Pentagons, is reducible into thirty-ſix Triangles. Now every Square equal to ABCD ſubtends ſix, as appears in the Figure; therefore the thirty-ſix Triangles are ſubtended by only ſix equal Squares, which compoſe a Cube inſcribed in the ſame Sphere; and therefore it may be truly ſaid that the diagonal of a Pentagon, *which is one of the Sides of a* Dodecaedron *inſcribed in a Sphere*, is equal to a ſide of a *Cube* inſcribed in the ſame Sphere.

161 Prob. 6. To *find the ſide of a* Dodecaedron, *and a Circle ſufficient to contain one of the Sides thereof.* Eucl. 13. Prop. 17.

In the firſt Place find the ſide of a Cube inſcribed in the propoſed Sphere:[‡] this ſide is equal to the diagonal of each pentagonal Face of the *Dodecaedron*:[§] divide this ſide into mean and extreme Ratio: its greater Part is the ſide of the *Dodecaedron* propoſed[‖] To find the Circle ſufficient to contain one of the Sides of the *Dodecaedron*, deſcribe a Pentagon as has been taught,[¶] then circumſcribe it with a Circle, and it will be that which was required.

162 Theor. 10. The *ſide of a* Dodecaedron *is incommenſurable with the diameter of the Sphere, both in itſelf and in the ſecond Power.* Eucl. 13. Prop. 17.

1. Let the diameter of the Sphere be b, that of the ſide of the Cube inſcribed in the Sphere be $c+d$, divided into mean and extreme Ratio, whoſe greater Part c is the ſide of the *Dodecaedron*.

---

[*] *Sup. n* 145.

[†] *B.* 2. *n.* 113.

[‡] *Sup. n.* 154.

[§] *Sup. n.* 160.

[‖] *B.* 4. *n.* 153.

[¶] *B.* 4. *n.* 159.

caedron. * 2. c is incommenſurable, with $c+d$ both in itſelf and in Power. † 3. $c+d$ is commenſurable with b in the ſecond Power, ‡ *that is,* $cc+2cd+dd$ is commenſurable with bb. cc is then incommenſurable with bb; for if it were commenſurable with bb, it would be ſo with the Square of $c+d$, § which bb is commenſurable to, ſeeing it is the Triple of the Square, ‖ conſequently cc incommenſurable in power with bb, is alſo incommenſurable in itſelf with bb.

LEMMA 2. *MN is the Diameter of a Circle, in which the two Chords AB and CE which interſect MN at right Angles, are Parallel one to another, and the Diſtance of FG equal to the half of each: I ſay that MF is the Side of a Decagon inſcribed in a Circle, whoſe Radius is FA.* 163

SUPPOSING MF or GN$=x$, and AF$=z+x$: if MF or x is the Side of a Decagon whoſe Radius is AF or $z+x$; then having divided AF into mean and extreme Ratio, x is the mean Part, ¶ and conſequently $\div\div\ z+x.\ x.\ z$; therefore if we demonſtrate that $\div\div\ z+x.\ x.\ z$ we ſhall do what is propoſed. Seeing AF$=z+x$; then AB$=2z+2x$, and BC$=z+x$. The Square of AB, which is $4zz+8zx+4xx$, with that of BC, which is $zz+2zz+xx$ are equal to that of AC or MN. † Now by the Hypotheſis MN$=3x+z$; for MF and GN are each equal to x, and FG$=z+x$: The Square, *I ſay* of $3x+z$ is $9xx+6xz+zz$: then putting the Squares of AB and BC both into one Sum, $9xx+6xz+zz=5zz+10zx+5xx$; taking away $5xx+6xz+zz$, from each part, there will remain $4xx=4zz+4zx$, dividing both by 4, it will be $xx=zz+zx$; then reſolving this Equation into a Proportion ‡ it will be $\div\div\ z+x.\ x.\ z$, ſeeing that the Product of the Extremes $z+x$ and z, which is $zz+zx$, is equal to xx the Square of the mean Term x; *Q. E. D.*

LEMMA. 3. THE *Line AM is the Side of a Pentagon inſcribed in a Circle, whoſe Radius is AF.* (ſame fig.) 164

MF is the Side of a Decagon inſcribed in a Circle, whoſe Radius is AF, § the Square of AF together with that of MF is equal to that of AM; therefore AM is the Side of a Pentagon. ‖

LEMMA 4. THE *Square of AF, or BF, or GE, or GC, which are equal Lines, is the fifth Part of that of the Diameter AC, or MN.* (ſame fig.) 165

LET AF$=b$; then AB$=2b$, and BC$=b$, the Square of AB is $4bb$, and that of BC is bb. Now both the Squares which make $5bb$, are equal to that of AC or of MN: *therefore it is equal to five times that of AF.

Lemma

* *B.* 4, *n.* 153. † *B.* 4. *n.* 140. ‡ *Sup.* *n.* 155.
§ *B.* 4. *n.* 113. ‖ *Sup.* *n.* 153.
† *B.* 4. *n.* 115. ¶ *B.* 4, *n.* 151. † *B.* 4. *n.* 78.
‡ *B.* 3. *n.* 58. § *Sup.* *n.* 163.
‖ *B.* 4. *n.* 161. * *B.* 4. *n.* 78.

166 **LEMMA** 5. TO *find a Line whoſe Square is the fifth Part of that of MN, the Diameter of the given Circle X.*

AM is the fifth Part of the Square of MN, the Square of MN is five times that of MB; † hence MB is the Line that was required, *that is*, equal to AF of the preceeding Figure, ſeeing the Square of AF is the fifth Part of MN. ‡

167 **PROBLEM** 7. *MN the Diameter of a Sphere being given, to deſcribe an Icoſaedron.* Eucl. 13. Prop. 16.

1. HAVING found the Length of AF, (*ſee the Figure of Lemma the 2d. Number* 163,) and having divided it equally in two: From the Center make DF and DG equal to the Half thereof, ſo that FG=AF; then draw AB and CE, interſecting MN at right Angles. 2. Taking AB and CE for Diameters, deſcribe two Circles which I call Z and Y, which are Parallel, (*ſee the following Figure,*) being upon Planes ſuppoſed to be Parallel. 3. In each of theſe two Circles inſcribe a Pentagon, and from each of its Angles draw right Lines to M and to N the Ends of the Diameter of the Sphere; which make five Triangles, whoſe Sides are each equal to the Side of the Pentagon inſcribed in theſe two Circles, *by Lemma the Third*; hence all the Sides of theſe Triangles being equal to the Sides of Pentagons, they form two ſolid Angles upon the Circles Z and X, each conſiſting of five Equilateral Triangles, whoſe Vertices are at the Extremes of M and N of the Diameter of the Sphere, and therefore here is found ten Sides of the *Icoſaedron.* Neither the Sides, nor ſolidAngle whoſe Vertex is N, are marked with Letters for fear of rendering the Figure confuſed, they muſt therefore be conceived by the Imagination. 4. Alſo in the ſame Circles Z and X, inſcribe a Decagon, whoſe coreſponding Angles in X and Z join by the Lines BE, PK, IH, DG, FL, &c. which by the Hypotheſis are all equal to the Radii of Z and X. 5. Draw the Diagonals BK, KI, IG, GF, &c. The Square BP, the Side of the Decagon, and that of PK, which is equal to the Radius of Z and X, are together equal to that of BK; § therefore BK is the Side of a Pentagon inſcribed in X and Z. ‖ The ſame Thing is demonſtrable of KI, of IG, of GF, &c. The Triangles BKI, KIG, ICF, &c. have for their Baſes the Sides of the ſaid Pentagon; they are therefore Equilateral one to another, and to the Ten which compoſe the two ſolid Angles, (*which we treated of above.*) conſequently there are between Z and X ten of theſe Triangles, five of which have their Baſes upon Z, and the other Five upon X, which with the Ten already found make the Twenty equal and equilateral Triangles which ought to conſtitute the *Icoſaedron*; *which was to be done.*

Corollary

---

† *B.* 4. *n.* 63. ‡ *Sup. n.* 165. § *B.* 4. *n.* 78.

‖ *B.* 4. *n.* 165.

168 COROLLARY 1. THE *Square of the Diameter of the Sphere is quintuple the Square of the Radius of the Circle X or Z, which is the Baſe of a ſolid Angle compos'd of five Equilateral ones: As has been demonſtrated in the Theorem, and elſewhere.* *

169 COROL. 2. THE *Diameter MN is compounded of the Side of the Hexagon, or Radii of the Circles Z and X, and of two Sides of the Decagon inſcribed in theſe Circles. As has been demonſtrated both in the Theorem and before.* †

170 COROL. 3. THE *Sides of the Triangles of the* Icoſaedron *are equal to the Sides of the Pentagons inſcribed in Z or X.*

171 THEOR. 11. THE *Sides of the* Icoſaedron *are incommenſurable, both in themſelves and in Power, with the* Diameter *of its circumſcribing Sphere.* Eucl. 13. Prop. 16.

THE Square of the Radius of the Circles which are deſcribed in order to make the *Icoſaedron*, is the fifth part of that of the Diameter of the Sphere. ‡ Let the Square be bb, therefore that of the Diameter of the Sphere is $5bb$. Theſe two Squares are then commenſurable, they being as 1 to 5. Let x be the Side of the Triangles which conſtitute the *Icoſaedron*, which x is one of the Sides of the Pentagon inſcribed in a Circle whoſe Radius is b, § wherefore bb and xx the Squares of the Side of the Pentagon whoſe Radius is b, are incommenſurable, as well as their roots x *and* b. ‖ and ſince that bb the Square of the Radius is commenſurable with the Square of the Diameter of the Sphere, then xx is incommenſurable with the Square of the Diameter; for if it were commenſurable thereto, it would be ſo with that of b; *and if xx and the Square of the Diameter are incommenſurable, x and the Diameter are ſo likewiſe, ſeeing the Ratios of Squares are duple that of their Roots, and that therefore if the Duples are Surds, then the compounding Parts are ſo likewiſe: For the Product of two numbers is a Number.

172 THEOR. 12. THE *ſame Circle comprehends the Pentagon which is one of the Sides of the* Dodecaedron, *and the equilateral Triangle which is one of the Sides of the* Icoſaedron. Eucl. 14. Prop. 3.

X and Z are two Circles, in which ſuppoſe the Pentagon of X to be one of the Sides of a *Dodecaedron*, and the equilateral Triangle of Z one of the Faces of an *Icoſaedron*, and theſe two Bodies inſcribed in one and the ſame Sphere whoſe Diameter is a. It muſt be proved that m, the Radius of X, is equal to n the Radius of Z. Deſcribe the Pentagon Y having each Side equal to the Side of the Equilateral Triangle of the *Icoſaedron*. The Pentagon is therefore the Baſe of five of the Triangles of the *Icoſaedron*. † Divide DF the Radius of Y, at the Point G into mean and extreme Ratio, The greater Part DG is the Side of the Decagon. ‡ Let BC be likewiſe divided into mean extreme Ratio at the Point K, the greater Part BK is equal to the Side of the Pentagon: * Therefore BC. DF::BK. DG; † hence $\overline{BC}^2.\ \overline{DF}^2::\overline{BK}^2.\ \overline{DG}^2$. ‡ And $3\overline{BC}^2.\ 5\overline{DF}^2::3\overline{BK}^2.\ 5\overline{DG}^2$. § Now BC is equal to the Side of the *Hexaedron*, ‖ three Squares of which are equal to *aa*, (*the Square of the Diameter of its circumſcribing Sphere*. * And 5DF is alſo equal: to *aa*† Then $3\overline{BC}^2 = 5\overline{DF}^2$. Hence there having been an Equality in the above Proportion, between $3\overline{BC}^2$ and $5\overline{DF}^2$. There is alſo an Equality between $3\overline{BK}^2$ and $5\overline{DG}^2$: Then ſeeing that $\overline{FE}^2 = \overline{DF}^2 + \overline{DG}^2$; ‡ then $5\overline{FE}^2 = 5\overline{DF}^2 + 5\overline{DG}^2$: Then alſo $5\overline{FE}^2 = 3\overline{BC}^2 + 3\overline{BK}^2$: For it has been ſhewn that $3\overline{BC}^2 = 5DF$, and $3\overline{BK}^2 = 5\overline{DG}^2$, but $3\overline{BC}^2 + 3\overline{BK}^2 = 15mm$, § and $5\overline{FE}^2 = 13nn$; ‖ For FE is ſuppoſed equal to the Side of the equilateral Triangle whoſe Radius is *n*, and conſequently ſince $15mm = 3\overline{BC}^2 + 3\overline{BK}^2 = 5\overline{FE}^2 = 15nn$. Therefore $15mm = 15nn$, or $mm = nn$, and conſequently $m = n$; *which was to be proved.*

---

* *Sup. n.* 165. † *Sup. n.* 163. ‡ *Sup. n.* 168. § *Sup. n.* 170. ‖ *B.* 4. *n.* 170.

* *B.* 4. *n.* 113. † *Sup. n.* 170. ‡ *B.* 4. *n.* 151.

173 PROB. 8. THE *Diameter of a Sphere being given, to find the Sides of the five regular Bodies which are inſcribed therein.* Eucl. 13. Prop. 18.

PROCEED as been taught, concerning the Finding of the Proportion of each Side of theſe regular Bodies to the Diameter of their circumſcribing Sphere.

174 THEOR. 13. THE *Superſicies of a* Dodecaedron *is equal to thirty times the Rectangle comprehended under one of the Sides of the Pentagon, which is one of its Faces, and the* Apotome *of that Pentagon; and that of the* Icoſaedron, *is equal to thirty times the Rectangle comprehended under one of the Sides of the equilateral Triangle, which is one of its Faces, and the* Apotome *of that Triangle.* Eucl. 14. Prop. 4.

HAVING drawn Lines from the Center of the Pentagon and from that of the equilateral Triangle to their angular Points, X will be thereby divided into five Triangles, and Z into three; hence the twelve Faces of the *Dodecaedron* into ſixty Triangles, as likewiſe the twenty Faces of the *Icoſaedron* into ſixty. Now each of theſe Triangles, as ABC. is equal to the Rectangle of AD its Apotome, and of BD the Half of BC, as is alſo GEH

* *B.* 4. *n.* 153. † *B.* 4. *n.* 67. ‡ *B.* 3. *n.* 84. § *B.* 3. *n.* 54. ‖ *Sup. n.* 160.

* *Sup. n.* 153. † *Sup. n.* 168. ‡ *B.* 4. *n.* 164. § *B.* 4. *n.* 165. ‖ *B.* 4. *n.* 146.

GEH to the Rectangle of EF by GF the Half of GH: * Therefore, &c.

COROLLARY. THEREFORE *the Surface of the* Dodecaedron *is* 175 *to that of the* Icoſaedron, *as* $AD \times BD$ *is to* $EF \times FG$.

LEMMA 6. THE *Apotome of a Pentagon is equal to the Half* 176 *of a Line equal to the Side of the Hexagon and of the Decagon inſcribed in the ſame Circle.* Eucl. 14. Prop. 1.

BC is the Side of a Pentagon, conſequently FC, the Side of half the Arc BFC, is the Side of the Decagon, and AC the Radius of the Circle, the Side of the Hexagon. AE is the Apotome of the Pentagon. It muſt be proved that AE is equal to the Half of $AC+CF$. Make GE equal to EF; and therefore $GC=FC$. † 1. FC being the tenth Part of the Circle, the Angle FAC is 36 Degrees; and ſince the Arc CD contains four times 36 Degrees, the Angle CFD is then of 72 Degrees, the Double of 36. ‡ Seeing that $GC=CF$: then FCG is Iſoceles, the Angle FGC is then alſo of 72 Degrees: CGA is then of 108. § Add GAC of 36, and it will be 144 which deduct from 180, there remains 36, the Content of ACG, which is therefore equal to GAC; conſequently $AG=GC=FC$: Therefore $AG=FC$. Hence $AG+GE=FC+FE$, or $AE=FC+FE$: Therefore $AE+EF+FC$ is the Double of AE: Then AE is the Half of $AE+EF$, *or the Radius* $AF+FC$; and conſequently half the Side of the Hexagon, and of FC the Side of the Decagon.

LEMMA, 7. *ABC is an equilateral Triangle, AE a Perpendicular which biſects BC; I ſay that* $DE=EF$. 177

BF is the Side of an Hexagon, therefore equal to the Radius BD; therefore $DE=EF$. ‖

THEOR. 14. THE *Surface of a* Dodecaedron *is to that of an* Icoſaedron, *as the Side of the Cube is to the Side of the* Icoſaedron, *both being inſcribed in one and the ſame Sphere.* Eucl. 14. Prop. 5. 178

AD is the Side of the Triangle which is one of the Sides or Faces of the *Icoſaedron,* and BD is the Side of one of the Faces of a *Dodecaedron* inſcribed in the ſame Circle. * EF equally divides AD, as EC doth BD; and conſequently the Arc BCD: † therefore CD is the Chord of the Decagon. H is the Side a Cube inſcribible in the ſame Sphere. 1. ∺ $EC+CD$. EC. CD. ‡ Now EG is the Half of $EC+CD$, § and EF the Half of EC, ‖ as is alſo $EG-EF$ the Half of CD: For $2EG=EC+$

A a +

---

* *B.* 2. *n.* 135. † *B.* 1. *n.* 58. ‡ *B.* 2. *n.* 59.
§ *B.* 2. *n.* 17. ‖ *B.* 1. *n.* 61.

* *Sup. n.* 172. † *B.* 1. *n.* 92. ‡ *B.* 4. *n.* 154.
§ *Sup. n.* 176. ‖ *Sup. n.* 177.

+CD, * and 2EF=EC: † then 2EG=2EF+CD. Taking away 2EF from each ſide, it becomes 2EG—2EF=CD; then EG—EF is the half of CD. It has been ſhewn that ∺ EC+CD. EC. CD: Then their Halves are in the ſame Ratio; therefore ∺ EG. EF. EG—EF. But the Line H being the Side of a Cube, if it be divided into mean and extreme Ratio, its greateſt Segment will be BD. ‡ Hence ∺ H. BD. H—BD. Then H. BD::EG. EF: § Then H×EF=BD×EG. Now H. AD::H×EF. AD×EF. ‖ Then H. AD::BD×EG. AD×EF; *that is,* as H, the Side of the Cube, is to AD, the Side of the *Icoſaedron,* ſo BD×EG is to AD×EF; but theſe two Products are to each other in the ſame Proportion as the Surfaces of the *Dodecaedron,* and *Icoſaedron:* * Therefore, &c. *Q. E. D.*

179 THEOR. 15. THE *Radius AB being divided into mean and extreme Ratio at C, and the greater Segment AC, the Square of BG the Side of the* Cube, *is to that of BK the Side of the* Icoſaedron, *as* $\overline{AB}^2+\overline{AC}^2$ *is to* $\overline{AB}^2+\overline{BC}^2$. Eucl. 14. Prop. 6.

LET BFGHI be a Pentagon and one of the Faces of the *Dodecaedron,* and BKL a triangular Face of the *Icoſaedron, (which they may be.)* † BG is the Side of a *Cube* inſcribed in one and the ſame Sphere: ‡ Then $\overline{BK}^2=3\overline{AB}^2$. § Now $\overline{AB}^2+\overline{CB}^2=3\overline{AC}^2$: * Then is $\overline{BK}^2$ triple $\overline{AB}^2$, as $\overline{AB}^2+\overline{CB}^2$ is triple $\overline{AC}^2$. Hence $\overline{BK}^2$. $\overline{AB}^2$::$\overline{AB}^2+\overline{CB}^2$ $\overline{AC}^2$ *Permutatively,* $\overline{BK}^2$. $\overline{AB}^2+\overline{CB}^2$::$\overline{AB}^2$. $\overline{AC}^2$. Now BF is the greater Segment of BG, *(the Side of the Cube divided into mean and extreme Ratio;)* † then $\overline{AB}^2$. $\overline{AC}^2$::$\overline{BG}^2$. $\overline{BF}^2$, ‡ ::$\overline{BK}^2$. $\overline{AB}^2\times\overline{CB}^2$. § *Permutatively,* $\overline{BG}^2$. $\overline{BK}^2$::$\overline{BF}^2$. $\overline{AB}^2+\overline{CB}^2$. Now $\overline{BF}^2=\overline{AB}^2+\overline{AC}^2$. ‖ and AB and AC being Sides of the Hexagon and of the Decagon; * $\overline{AB}^2+\overline{AC}^2$ being put in the Place of $\overline{BF}^2$, we ſhall have $\overline{BG}^2$. $\overline{BK}^2$::$\overline{AB}^2+\overline{AC}^2$.
180 $\overline{AB}^2\times\overline{CB}^2$; *which was to be proved.*

---

* *Sup. n.* 176. † *Sup. n.* 177. ‡ *Sup. n.* 160. & *B.* 4. *n.* 153. § *B.* 4. *n.* 67. ‖ *B.* 4. *n.* 75.

* *Sup. n.* 175. † *Sup. n.* 172. ‡ *Sup. n.* 160. § *B.* 4. *n.* 146. * *B.* 4. *n.* 70.

† *Sup. n* 160. ‡ *B.* 4. *n:* 67. § *B.* 3. *n:* 53. ‖ *B.* 4. *n.* 164. * *B.* 4. *n.* 151.

THEOR. 16. As *the Side of the* Cube, *is to the Side of the* Icoſaedron, *ſo is the* Dodecaedron *to the* Icoſaedron *both being inſcribed in the ſame Sphere.* Eucl. 14. Prop. 7. 180

CONCEIVE right Lines to be drawn from all the Angles of the *Dodecaedron*, and likewiſe of the *Icoſaedron*, to the Center of the Sphere; which will make in one, twelve Pyramids, and in the other twenty, which are of the ſame Height, ſeeing that the ſame Circle of the Sphere contains one of the Faces both of the *Dodecaedron* and of the *Icoſaedron*; * therefore theſe Pyramids are as their Baſes, † *that is*, as the Surfaces of theſe two Polyedrons. Now theſe Surfaces are one to another ‡ as the Side of the *Cube* is to the Side of the *Icoſaedron*.

ADVERTISEMENT.

WHAT follows will be eaſily perceived, if what is here mentioned, be actually done upon the Sides or Faces of thoſe Solids; whoſe true Forms are exhibited in Paſte-board, Plate 1.

PROB. 7. To *inſcribe a* Tetraedron *in a* Cube. Eucl. 15. 181
Prop. 1.

FROM A one of the Angles of the Cube X, imagine the Diagonals AB, AC, AD, and BD, BC, CD to be drawn, and you will perceive a *Tetraedron* ABCD or Pyramid compoſed of four equilateral Triangles, for the Diagonals are equal; therefore the Triangle ABC=BAD, &c.

PROB. 8. To *inſcribe an* Octoedron *in a Pyramid or* Tetraedron. Eucl. 15. Prop. 2. 182

DIVIDE all the ſix Sides of the Pyramid or *Tetraedron* equally in two; join the Points of Section by twelve Lines which are all equal, and make eight equilateral Triangles.

PROB. 9. To *inſcribe an* Octoedron *in a* Cube. Eucl. 15.
Prop. 3. 183

HAVING found the Center of each Side of the Cube, join them by drawing as many Lines as are requiſite, *viz.* twelve which being equal, they make eight equal Triangles, and conſequently an *Octoedron*.

PROB. 10. To *inſcribe a* Cube *in an* Octoedron. Eucl. 15 184
Prop. 4.

1. DIVIDE all the Sides of the *Octoedron* into two equal Parts. 2. Draw Lines thro' all thoſe Sections upon the Sides of the Body, theſe Lines are all equal, and make two oppoſite Squares, which being joined by four other Lines, which are equal to the firſt Lines, and the oppoſite ones being Parallel, they make a Cube; *which is evident.*

---

* *Sup. n.* 172. † *Sup. n.* 123. ‡ *Sup. n.* 198.

185 Prob. 11. To *inſcribe a* Dodecaedron *in an* Icoſaedron. Eucl. 15. Prop. 5.

Five of the *Icoſaedron's* Triangles which make a ſolid Angle or Pyramid, have for their baſe a Pentagon. § Divide therefore all the Sides of this Pyramid equally in two ; the Plane of this Section will be a Pentagon and one of the Faces of the *Dodecaedron*; for by doing the ſame to all the ſolid Angles or Pyramids of the *Icoſaedron*, which are 12 in number, there will be twelve equal Pentagons, which conſtitute the twelve Faces of the *Dodecaedron*.

ADVERTISEMENT.

The Diameter of the Sphere being ſuppoſed 1000, the Sides of the regular Bodies inſcribed therein, are nearly as the following Numbers.

The *Tetraedron*	816
The *Octoedron*	707
The *Hexaedron* or *Cube*	577
The *Icoſaedron*	527
The *Dodecaedron*	357

Recreative Problems.

186 Problem. 1. To *prolong a Line which is too ſhort, Geometrically.*

Altho' in Euclid this be taken for a Principle, yet in practice when the given Line is very Short, it will be difficult to perform it exactly, by the bare Application of a Ruler. Let the given Line be AB, with the Extent thereof and one foot in A, deſcribe thro' B the Arc CBD, from which take equal Arcs, as BC, BD, from C and D with an equal Extent deſcribe two Arcs, and the Point where they interſect will be in a right Line with AB, thus may the Line given be prolonged; *as is evident.*

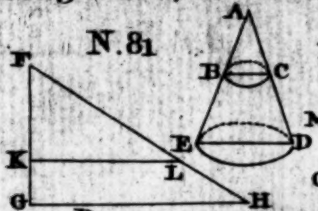
13 When *there is a Wall as CD, or any other Impediment at the End of the given Line, proceed thus.*

CD is a Wall at the End of the given Line AB, required to be prolonged from its End B. From B let fall a Perpendicular BE, at right Angles with AB, and of any length you pleaſe, draw Lines from the End thereof to any two Points in AB, as EF and EA, then make the Angle BEG equal to BEF. and the Line EG equal to the Line EF; alſo make the Angle BEH equal to the Angle BEA, and the Line EH equal to EA: then will a Line drawn thro'

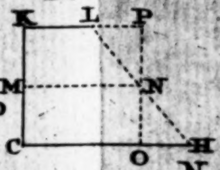
---

§ *Sup. n.* 167.

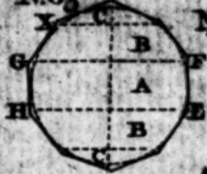
N.81



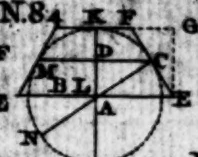
N.82



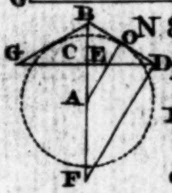
N.83



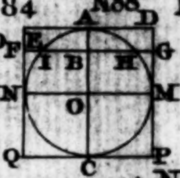
N.84



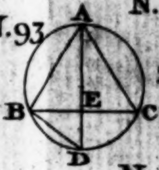
N 84



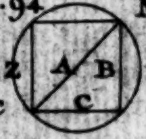
N88



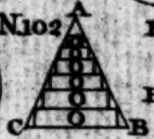
N.93



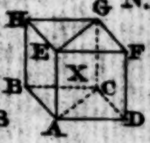
N.94



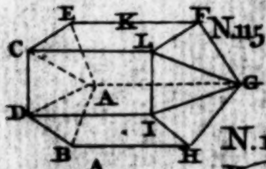
N.102



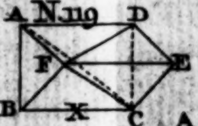
N.105



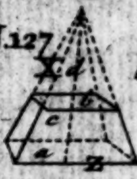
N.115



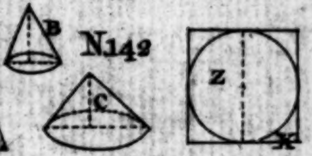
N.119



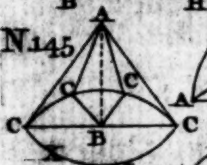
N.127



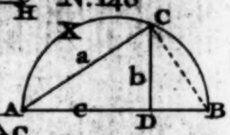
N.142



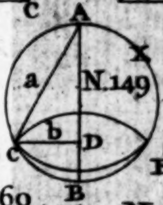
N.145



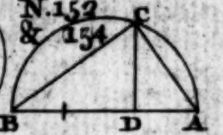
N.148



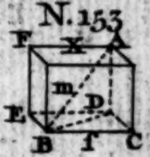
N.149



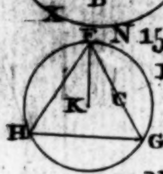
N.152 & 154

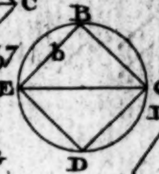


N.153

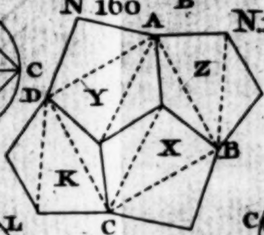


N 157

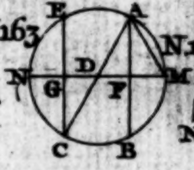




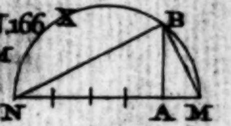
N 160



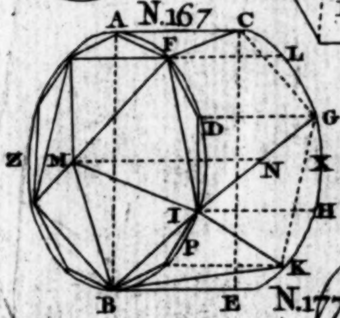
N.163

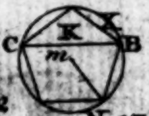


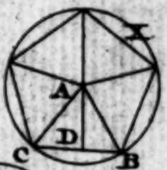
N.166



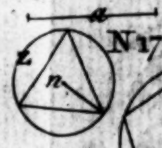
N.167



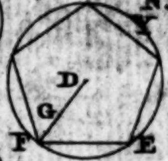


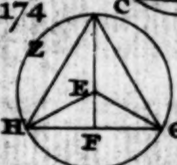


N.172

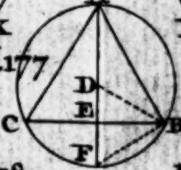


N.174

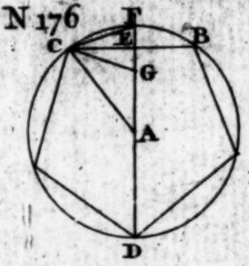




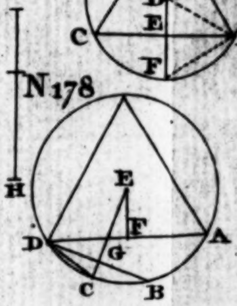
N.177



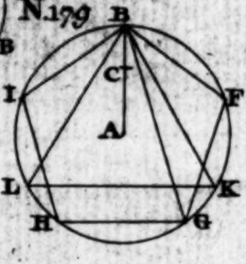
N 176



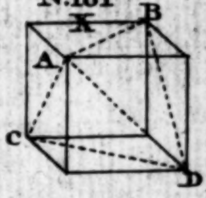
N 178



N.179



N.181



thro' G and H be in a right Line with AB; *which was required.*

PROB. 2. To *tranſform a given Triangle, into another equal* 188
Triangle, having each Side thereof longer than each Side of the Triangle given.

LET the given Triangle be ABC, prolong its Baſe AB on both Sides to D and E, making AD equal to AC, and BE equal to BC; then draw at right Angles to DE, a Line equal to the fourth Proportional between DE, and AB, the Baſe of the given Triangle, and CH the Height thereof, ſuch is EG or DF, and compleat the Parallelogram DEGF, then draw Lines from F and G, to any point of the Line DE which is between A and B, and they will form the Triangle required.

SCHOLIUM. IF the Lines drawn from F and G do not reach to the Line DE, but join at ſome Point within the Parallelogram, and under the Line AB, they will form a Triangle whoſe Content is leſſer than that of the given one, and all its Sides longer.

PROB. 3. To *divide the Circumference of a given Semi-Circle,* 189
into two ſuch unequal parts, that the Semi-Diameter, or Radius, may be a mean Proportional between the Chords of thoſe two Arcs.

LET the given Semi-circle be ABC, its Center D. From B the End of the Diameter AB, deſcribe thro' D the Arc DE, and having divided the Arc BE into two equal parts at C, draw the Chords AC, BC, between which the Semi-Diameter AD or CD is a mean Proportional.

PROB. 4. To *deſcribe two different Iſoceles Triangles, having* 190
each the ſame Area, and the ſame Circumference.

Draw AB, equal to 24 on a Scale of equal parts, and DE=40, upon the Middle of each draw the Perpendiculars CG=35, and FH=21, draw CA. CB and FD, FE, then will the Area and Circumference of the Triangle ACB be each equal to thoſe of the Triangle DFE.

PROB. 5. To *deſcribe three different right angled Triangles,* 191
having equal Areas.

Draw for the Baſe of the Firſt the Line AB=42, and the Perpendicular AC=40: And draw the Hypothenuſe BC which is equal to 58 on the ſame Scale. Then draw DE=70, for the Baſe of the Second, and the Perpendicular DF=24; then will the Hypothenuſe FE=74. Laſtly draw FG=112, and the Perpendicular EH=15. Then is the Hypothenuſe GH=113. Then proceeding with thoſe numbers according to *Book* 2. *n.* 135. It will appear that the Areas of the Triangles CAB, FDE and HFG are equal one to another.

PROB.

192 PROB. 6. To *deſcribe three equal Triangles, the Firſt of which is right Angled, the Second an Oxygonium, and the Third an Ambligonium.*

Draw AB=24, for the Baſe of the right angled Triangle, and AC=7, then is the Hypothenuſe BC=25. *Secondly*, For the Baſe of the Second draw DE and take therefrom the Segment KD=5, and KE=9, and from the Point K raiſe the Perpendicular KF=12; and compleat the Triangle by drawing the Lines FD and FE. *Thirdly*, Draw GH for the Baſe of the Obtuſe angled Triangle, and make the Segment LG=6. and LH=15, Erect the Perpendicular LI at the Point L equal to 8; and draw the Lines IG and IH which form the Triangle required: then are thoſe three different Triangles equal to each other; as was propoſed.

193 PROB. 7. To *deſcribe a right angled Triangle, having its three Sides in Geometrical Proportion.*

WITH any Extent deſcribe a Semi-circle as ABC, whoſe Center is D, take the Diameter AC for the Hypothenuſe of the Triangle required: From C, the Extremity of AC, draw the Perpendicular CE, equal and Perpendicular to the Diameter AC, and draw DE, which cuts the Circumference at F. Take the Diſtance of FE and and ſet it off upon the Circumference from A to B, and draw the Lines AB, BC, then is ABC right angled at B, and its Sides are in the required Ratio.

194 PROB. 8. To *deſcribe a right angled Triangle, having its three Sides in Arithmetical Proportion.*

DRAW an indefinite Line and mark thereupon five equal parts, as here from A to B; and account AB the Hypothenuſe of the Triangle required. From A, with the Extent of three of thoſe equal parts, draw an Arc, and from B, with the Extent of four parts, draw another Arc, which will interſect the Firſt at C, draw the Lines AC and BC; and they will form the Triangle required.

195 PROB. 9. Two *Lines being given Perpendicular to a Line drawn thro' their Extremities, to find upon that Line a Point equally diſtant from each of the two other Extremities.* Or, *the Breadth of a Street being given, to find in what part thereof a Ladder of a determinate Length muſt be placed, ſo that its Top may reach from thence to two Windows of different Heights, on the oppoſite Sides of that ſame Street.*

LET AB be the Breadth of the Street, BC the Height of one Window, and AD the Height of the other, join D and C by the right Line DEC, from E the Middle thereof let fall the Perpendicular EF upon the Line AB, then is the Point F the Place required, the Line FC being equal to FD.

Suppoſe $AB=49$, $BC=56$, $AD=63$, then $AF=16$, $FB=33$, and FC and FD each equal to 65.

PROB. 10. To *deſcribe in a given Circle four equal Circles mutually touching each other, and likewiſe the Circumference of the given Circle.* 196

LET the given Circle be ABCD, its Center E, draw the Diameters AC, BD Perpendicular to each other, take DF equal to CD, the Chord of Ninety, then ſet EF from A to K, from B to G, from C to H, and from D to I; and then will the Points K, G, H, I, be the Centers of the Circles required.

PROB. 11. To *deſcribe four equal Circles which mutually touch each other, and alſo the Outſide of the Circumference of a given Circle.* 197

THE given Circle is ABCD, in which draw at right Angles the Diameters AC, BD, which interſect at the Center E; prolong the Diamer AC and take AF equal to AB, the Chord of Ninety, then is EF the Radius of each of the Circles required.

PROB. 12. To *deſcribe ſix equal Circles which mutually touch one another, and alſo the three Sides, and three Angles of an equilateral Triangle given.* 198

LET ABC be an equilateral Triangle given, whoſe Center is D, Biſect each Angle equally in two and draw the Lines AF, CE, BG, of a convenient Length. Upon the Side AB take EH equal to half the Perpendicular DE, draw DH, and prolong it making $HI=HE$, then is the whole Line DI the Radius of each Circle, which being ſet off upon the produced Lines from E to K, B to L, F to M, &c. will give the Centers required.

PROB. 13. To *deſcribe within an equilateral Triangle three equal Circles which touch one another, and alſo the three Sides of the equilateral Triangle.* 199

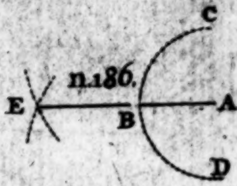
LET ABC be an equilateral Triangle, divide each of its Sides into two equal parts at the Points D, E, F, and thro' theſe Points draw Lines to the oppoſite Angles, then take the Extent of AD or DB, the Half of the Side of the given Triangle, and ſet it upon the Line AE from E to H, and aud upon the Line BF from F to I, and upon the Line CD from D to G, then are the Points G, H, I, the Centers of the Circles required: theſe Points being joined by right Lines, will form another equilateral Triangle having its Sides parallel to the Sides of the Triangle given.

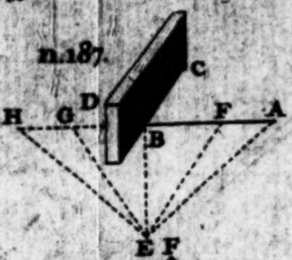
PROB. 14. To *inſcribe in a given Square, four equal Circles which touch one another, and likewiſe the Sides of the Square.* 200

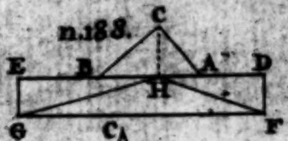
Divide each Side of the Square ABCD, into two equal Parts at the Points F, G, H, I, and draw the Lines FH, GI, which cut one another at right Angles in two equal parts at E, the Center of the Square, join the Points H, I, and take therefrom the Part IK=IE=EG, the Remainder KH is the Radius of each Circle, which ſet off upon the Lines FH, GI, from the Extremities thereof, and it will mark out the Center Points L, M, N, O.

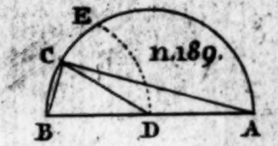
⁂

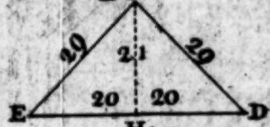
BOOK VI.

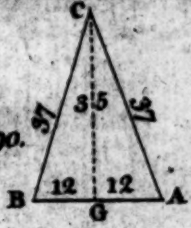

n.186.

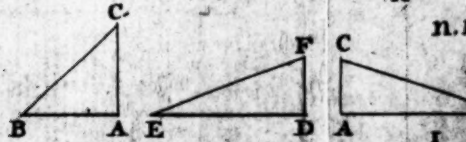

n.187.

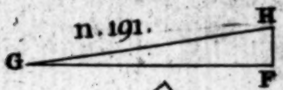

n.188.

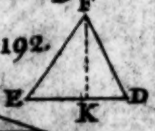

n.189.

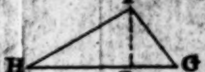


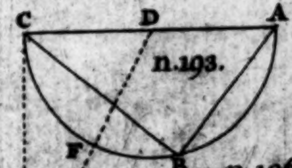

n.190.

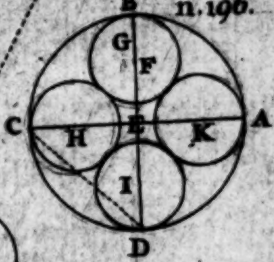


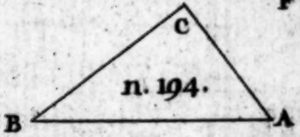

n.191.

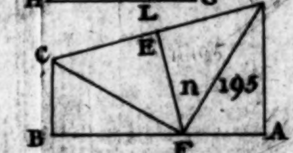

n.192.

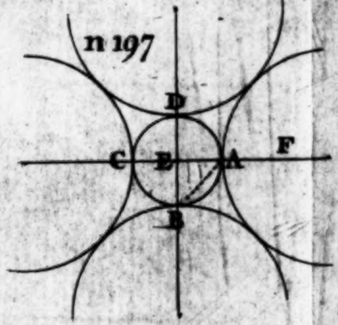


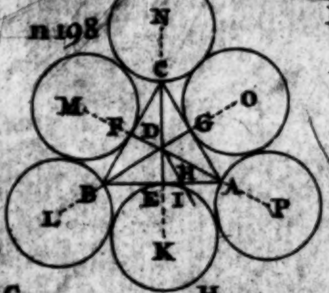

n.193.

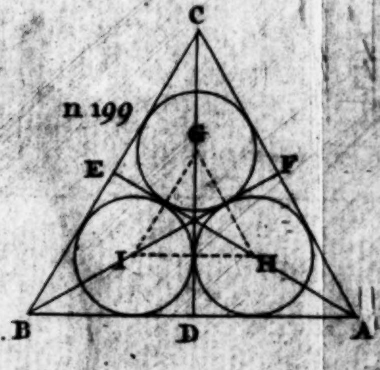

n.196.

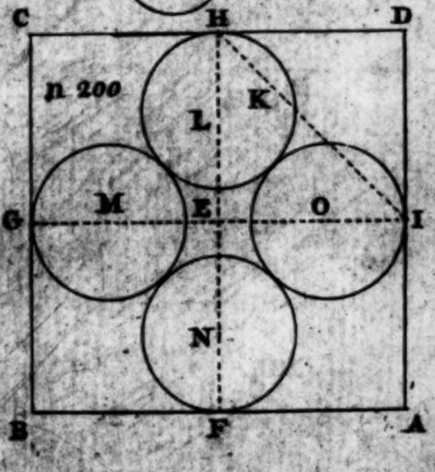

n. 194.


n 195


n 197


n 198


n 199


n 200

BOOK. VI.

OF METHOD.

ADVERTISEMENT.

THERE are two ways or methods of Solving Problems. The First, which proceeds from known or given Quantities, to the Discovery of the Thing required, has been chiefly made uſe of in the foregoing Part of this Treatiſe, and is commonly called the *Method of Doctrine*, whereas the Second, which is more Univerſal, begins with what is unknown and proceeding therewith as if it had been known, it is at laſt found equal to ſome known or given Quantities, and from thence its true Value is diſcovered: This is called the *Method of Invention*, and has been made uſe of heretofore, on certain occaſions, in order to find what was unknown, and which we ſhall now proceed to apply to the Solution of ſeveral uſeful Geometrical Problems.

CHAPTER I.

OF *the Examination of a Queſtion, or Problem. In the firſt Place, it muſt be well underſtood, and exactly expreſſed.*

THE Truth often lies ſo concealed in Problems whoſe Solutions are ſought, as not to be diſcovered by the bare application of the Mind: The Imagination muſt therefore be aſſiſted by drawing ſuch a Figure as is proper to expreſs thoſe things which are required to be conſidered. And this may be done, altho' ſome of thoſe things are unknown to us, but ſomething muſt be given otherwiſe it is impoſſible. Now from that which is given, we deduce ſuch Suppoſitions as the Nature of the Problem requires: Then drawing Lines according to what was ſuppoſed; thoſe Lines will be known, as will appear farther on, and from thence we diſcover thoſe which are required: Therefore the firſt thing that ought to be done, is to form ſuch a perfect Idea of the Queſtion, as to be able to expreſs and diſtinguiſh exactly thoſe 1 things which are to be found, and to free it from every Superfluity which ſerves only to render it obſcure. The following Examples ſhew that difficult Queſtions are often ſolved by the bare Expreſſion thereof. I here mean figural Expreſſions as well as Verbal ones.

QUESTION I.

2 To *Demonſtrate that the Surface of a Triangle is equal to half the Sum of its three Sides, multiplied by the Radius of its inſcribed Circle.*

The bare Sight of this Figure demonſtrates that this is true. Having drawn Lines from the Angles of the Triangle BCD, to A the Center of the inſcribed Circle, they make three Triangles BAC, CAD, DAB, together equal to the Triangle BCD; and whoſe height is the Radius of the Circle; they are therefore equal to the three Sides of BCD, and whoſe height is the Radius of the inſcribed Circle: * the Surface of the Triangle is therefore equal to the Product of half its Baſe by its Height; *which was to be proved.*

Also ſee here by another Example, how the Manner of expreſſing a Queſtion by a ſuitable Figure, facilitates the Reſolution.

3 Quest. 2. To *demonſtrate that in the Triangle ABC, if a Perpendicular be drawn from the Angle CAB upon BC, the Sum of the two Sides, AB and AC is to BC,* (the Baſe of the Angle which thoſe two Sides contain,) *as the Difference between CD and BD is to that between AC and AB.*

From A as a Center, and with the Extent of AB, the ſhorteſt Side deſcribe a Circle, and ſince AB is equal to AE, the Line CE is the Sum of the Sides AC and AB. The Lines AF and AB are equal, therefore CF is their difference. Alſo ſeeing DB=DG, the Line GC is the Difference between CD and DB. Hence here is an expreſſion, or Figure which diſcovers what is ſought. Therefore the Queſtion is now eaſy to be ſolved; for CE×CF=BC×CG: † Therefore CE. CB::CG. CF; ‡ Q. E. D.

C H A P. II.

4 ALTHO' *the Lines and Quantities concerned in a Queſtion are unknown, yet they may be expreſſed, and all the Operations of Arithmetic performed upon them.*

---

* B. 2. n. 143. † B. 4. n. 57. ‡ B. 3. n. 58.

THE Method which we now treat of is Univerſal, and very different from that which hath been uſed hitherto. The Buſineſs is here, to find and demonſtrate the Properties of the Figure which is in Queſtion; and in this Perſuit, we conſider Quantities only as being to be added, Subtracted, Multiplied, or divided. And how this may be done upon every thing that is capable of being augmented or diminiſhed, whether Lines or Numbers is ſhewn in Book the Third. hence this Method is already partly explained; therefore if the preceeding Books be read, we cannot then be ignorant that by $a+b$, may be underſtood two Lines a and b added together; by $a-b$, that b is ſubtracted from a: by ab that a hath been multiplied by b, or b by a: that therefore ab is a Rectangle, whoſe Sides are a and b; that aa, or $\overset{2}{a}$, is a Square, one of whoſe Sides is a; that $\frac{ab}{b}$ is a Rectangle, or Plane divided by b, after which Diviſion there remains only the Side a, *that is*, a Line only. According to what was taught at the beginning of the third Book, the Letter which is both above and below the Sign of Diviſion may be cancelled; and that therefore $\frac{ab}{b}=a$. Powers may likewiſe be expreſſed by Letters as well as by Lines; the Proportions of Figures may therefore be denoted by Letters; for if a Triangle is right angled, and its Hypothenuſe be a, and the two Sides b and c, according to what has been taught, * $aa=bb+cc$, and $aa-bb=cc$, and $aa-cc=bb$. 5

The Proportions of ſimilar Triangles may likewiſe be all expreſſed by Letters; for if ABC and DEF are ſimilar, that AB $=a$, and BC$=b$, and DE$=m$, and EF$=n$; then $a.\ b::m.\ n.$ and ſeeing that according to what has been ſaid, † b multiplied by m, and the Product bm divided by the firſt Term a, the Quotient $\frac{bm}{a}$ will be the fourth Term; Then EF$=\frac{bm}{a}$, and by the ſame Reaſoning $\frac{bm}{n}=$ AB. Which ſhews how to denote the Proportion of Lines, as well as thoſe of Numbers. This univerſal Method of expreſſing Quantity, and performing Arithmetical Operations thereupon, is called *Algebra*, which, tho' it be difficult to thoſe who have not been accuſtomed thereto, it will not be ſo to ſuch as make uſe of theſe Elements. Obſerve that the Product of a Multiplication may be reduced to a Proportion; for Example, the Product of a by b, to a Proportion whoſe firſt Term is unit, the Second and Third are the two Quantities a and b, which multiply together; and the fourth Term will be ab, which is the Product of the Multiplication, which 6 which is evident, For let the fourth Term be called x: Then $1.\ a::b.\ x.$ Now $\frac{ab}{1}=x.$ [*] But Unit neither divides nor multiplies. therefore $\frac{ab}{1}=ab,$ and therefore $ab=x.$ Unit may therefore be ſuppoſed tho' it be not expreſſed. Hence the Product of three Letters, as abc, may denote theſe two Proportions. $1.\ c::ab.\ abc$; and $1.\ ab::c.\ abc.$ This ſhews that the Product of four Lines $abcd$, may denote three Proportions, the Product of five Lines $abcde$, denotes four Proportions; and in theſe Proportions the whole Product is only a Line, which proceeds from all theſe Proportions, when the firſt Term is Unity; for ſeeing that $1.\ a::b.\ ab$; then $1.\ b::a.\ ab.$ likewiſe, ſince $1.\ ab::c.\ abc$; then $1.\ c::ab.\ abc.$ The Quotient of Diviſion expreſſes a Line, for Example, $\frac{a}{b}$: For it may be reduced to this Proportion $b.\ a::1.\ x.$ for $\frac{a1}{b}$ or $\frac{a}{b}=x.$

---

* *B.* 2. *n.* 142. † *B.* 3. *n.* 60.

8 The Operations of Arithmetic are all eaſily performed by the
Ruler and Compaſſes, one Line may be added to another, or
a Leſſer be ſubtracted from a Greater. To multiply one Line
by another, *that is,* to find a Line equal to the Product of two
Lines; proceed thus. Let the Lines given to be Multiplied be
AD and AC; upon AC take the Line AB equal to Unit, and
draw a Line AD making any Angle with AB. Draw a Line
thro' B and D, and thro' C another Parallel thereto, which being
done, AE is the Line required: For AB. AD::AC. AE; then
AB×AE=AD×AC. A Quantity being multiplied by only
Unit continues the ſame; therefore AE multiplied by AB,
which is a Unit; continues equal to the ſame as before, therefore
AE, which is a fourth Proportional, is equal to the Product
of AD by AC; *which was ſought.* If it be required to divide
9 AE by AC, having taken AB equal to Unit, and drawn
thro' B, a Parallel to CE, the Line AD is the Quotient of AE
divided by AC; for AB. AD::AC. AE, and Unit is to the
Quotient of a Diviſion, as the Diviſor is to the Dividend. If
10 it were required to extract the ſquare Root of GH, add thereto
a right Line FG, equal to a Unit, and divide FH into two equal
parts at the Point E: From the Center E deſcribe the Circle
FIH. Erect a Perpendicular from the Point G, and the Line
GI, is the Root required; for ∺ FG. GI. GH. Then the Square
of GI, is equal to the Product of GH. Now FG being Unit,
if GH be multiplied thereby, it will yet continue the ſame,
therefore GH is equal to the Square of GI, which is conſequently
the Root of GH, the ſame Way may be found a Line
equal

---

[*] *B.* 3. *n.* 60.

equal to the ſquare Root of a ſurd Number; for Example, equal to the ſquare Root of 18: for taking GH equal to 18, and adding thereto FG equal to unit; and from E the Middle of that Line as a Center, deſcribe a Circle, then is the Line GI equal to the ſquare Root of 18, which as we have ſhewn in the fourth Book, is not expreſſible by any number.

CHAP. III.

AFTER *having expreſſed a Queſtion or Problem; and deſcribed the Figure proper thereto, it muſt be conſidered whether the* 11 *Problem be determinate or indeterminate; and that which is known muſt be diſtinguiſhed from what is unknown.*

THE Problem being exactly conceived, and the Figure deſcribed which expreſſes its conditions, there muſt be diſtinguiſhed theſe three things. 1. The known Lines. 2. The unknown ones. 3. All the Proportions which the known and unknown Lines bear one to another. Some of the Lines which are drawn to expreſs a Problem are known, the Reſt are ſuppoſed, *that is,* 'tis ſuppoſed that they ought to have ſuch and ſuch Proportions with the known Lines. We ſhall here repreſent the known Lines by the firſt Letters of the Alphabet, calling them *a*, or *b*, or *c*, &c. and the unknown ones by the Laſt, *x*, *y*, *z*. It may ſometimes be proper to uſe the firſt Letter of the Names of the known and unknown Quantities, for Example, to repreſent a number by *n*, a Sum by *ſ*, time by *t*, velocity by *v*. Next conſider whether the Problem be ſo propoſed, as that the drawing certain Lines, depends upon the Choice of certain Quantities at pleaſure, or whether all thoſe which the Queſtion contains, are determined, ſo that all which can be done, be only to give them ſuitable names. It muſt then be immediately examined whether the Queſtion be determined, or indetermined. In an indetermined Queſtion, there appears no Proportion, which gives room to expreſs the unknown Quantities two ways, for if I can call the ſame Quantity either x, or $y+a$. 'tis becauſe I know that x is equal to y, after a is added thereto; that $x=y+a$, this is a determined Proportion. But when, without determining 12 any, it is for Example propoſed to divide the Diameter AB, ſo that the Rectangle of the two Parts may be equal to a Square, the Problem is indetermined: and I cannot find a double Expreſſion, but by ſuppoſing a certain Proportion; as this for Example, that AB is divided in C. After which, having deſcribed a Circle with the Extent of half AB, and upon C raiſed the Perpendicular CD, I ſhall have what was required.

∺ AC. CD. CB: * Then $AC \times CB = \overline{CD}^2$. † An indeterminate Problem admits of ſeveral Solutions; for what part ſoever of the Diameter C is in, the Square of CD will be equal to the Rectangle of the Parts of the Diameter if divided as ſuppoſed to be. AB may be divided in many places beſides C.

13 A Problem is ſaid to be determinate, when it cannot be Solved but by obſerving certain things which limit it, and which depend not upon the Will of the Propoſer, *that is,* the Quantities contained therein, have a certain Proportion peculiar to themſelves. For Example, AB being divided in C; it is propoſed to produce AB even to D, *(an unknown point;)* ſo that the Square of CD may be equal to the Rectangle of AD and BD: then as the prolonged part BD is determinate, *that is,* that it hath a certain exact Length; the Problem is determinate: And ſuppoſe that $AC=a$, and $CB=b$, and $BD=a$, there will be this double expreſſion, *viz.* For the Rectangle, $ax+bx+xx$, and for the Square $bb+2xb+xx$; which two Quantities ought *(according to the Suppoſition)* to be equal; therefore $ax+bx+xx=bb+2bx+xx$, hence may the value of x be diſcovered.

CHAP. IV.

14 THE *Proportions which are between the Lines of the Figure by which a Problem is expreſſed, being known, thence ariſes the Method of equalizing them, or making double expreſſions thereof, which is called an* Equation.

If it be known that three given Lines a, b, c are in Proportion, and that the unknown x is the fourth Term of that Proportion; that therefore $a.\ b :: c.\ x$, ‡ then ſeeing that $\frac{bc}{a}=x$, there is this double expreſſion of the ſame Quantity, $\frac{bc}{a}$ and x, and this is what is called an *Equation.* To find Equations, is to find double Expreſſions of an unknown Quantity; which may be done, when ſome of its Proportions with the given Quantities are known. If it is known that the Sum of a and b is equal to the unknown Quantity x, this Expreſſion $a+b=x$ is an Equation. It is evident *that the Half of two unequal Lines, leſs half their Difference, is equal to the leaſt Line; and that ſame half, more half their difference, is equal to the greateſt Line.*

Let the Line $AC=x+z$, $AD=x$, and $DC=z$, let their difference be DE, whoſe half is BD. Let $AD=y$, and $DE=a$, hence DB or BE$=a$, it is plain that AD–DB, or $y-a=$AD or x, and that BC+BD, or $y+a=$DC or z. Hence when the Difference between two unknown Quantities is known, they may be expreſſed two different ways.

---

* *B.* 4. *n.* 29. † *B.* 3. *n.* 57. ‡ *B.* 3. *n.* 60.

The Proportions which are between the Lines of a Geome- 16
trical Problem, are often diſcovered by means of the Figure it-
ſelf, which the Problem is expreſſed by; for if for Example,
I know the unknown Line x, to be the Hypothenuſe of a right
angled Triangle, whoſe two other Sides are d and b, I
know that $xx=bb+dd$; * hence here is a double Expreſſion of
x. A Plane is produced by the Multiplication of its two Roots;
if it be therefore divided by one of them, the Quotient of the
Diviſion is the other. Thus if $b.c::d.x$, ſince that $bx=cd$,
dividing cd by b, the Quotient $\frac{cd}{b}=x$. If x were a mean Pro-
portional between c and d, then $xx=cd$. That which was deli- 17
vered in the third Book concerning Powers, is alſo a Fund of dif-
ferent Equations: For if I know that $a+b=x$, then $aa+2ab$
$+bb=xx$. † If x is the Hypothenuſe of a Triangle, and that
the two Sides are a and b, then $aa+bb=xx$: wherefore $xx-$
$aa=bb$, or $xx-bb=aa$. The ſame Quantities may be diffe- 18
rently expreſſed, by denoting either the Parts for the Whole, or
the Whole for the Parts, the Powers for the Roots, or the Roots
for the Powers, for if $a+b=x$, then $aa+2ab+bb=xx$; or
$aa+2ab=xx-bb$, or $aa+bb=xx-2ab$. It is evident that an
infinite number of Equations may be found this way: if $ab=xx$;
then $\sqrt{ab}=x$.

CHAP. V.

19 THERE *muſt be found as many Equations as there are unknown Lines, and all theſe Equations muſt be reduced to one only.*

The main Scope or Deſign in a Queſtion, or Problem, is to 20
bring back, (*If I may ſo ſay*) the unknown Quantity, and to
compare it with the known, ſo as to obſerve their Difference,
thence is diſcovered what muſt be added or ſubtracted, in order
to render them equal; which may alſo be done when their Ra-
tio is exactly known: For if x is the third Part of a, then a
$=\frac{x}{3}$. If 'tis a which is third Part of x, then $3a=x$. Thus
may Equations or double Expreſſions be found for all the Quan- 21
tities in the Problem; and there muſt be as many found as there
are

---

* *B*. 4. *n*. 79.

† *B*. 3. *n*. 20.

are unknown Quantities. There is always in a Problem one principal unknown Line, which the Solving of the Problem depends upon: and 'tis to that Equation, that all the reſt muſt be reduced, ſo that there may be only one unknown Letter; for if it is known that $x+b=z$, or that $z-b=x$, wherefore where thoſe two unknown Quantities are, I may always ſubſtitute $x+b$ in the Room of z, or $z-b$ in the Place of x, and conſequently reduce the Queſtion to other Terms having only one unknown Quantity. *There are different Methods for doing this.* When the unknown Quantity is known to be the third or fourth Term of a Proportion, it may be expreſſed by the Letters of the known Quantities; for ∺ $a.\ b.\ x$; then $\frac{bb}{a}=x$.* If $a.\ b::c.\ x$; then $\frac{bc}{a}=x$. The Terms of a Progreſſion may expreſſed ſo as to make uſe only of the ſame Letter. If this progreſſion of four Terms be expreſſed by four different Letters ∺ $b.\ x.\ y$. It may be reduced ſo that there remain but one unknown Letter; for ſuppoſe $b=1$, I may then ſay that ∺ $1.\ z::zz.\ zzz$. $1.\ b$ or 1 is to x, as the Square of b or 1 is to that of z, † *that is*, $b.\ x::bb.\ zz$, or $1.\ x::1.\ zz$; hence ſince $zz=x$, I may ſubſtitute zz in the Place of x. So likewiſe b or 1 is to y, as bbb or 1 is to zzz, ‡ then as $zzz=y$, inſtead of y, I put zzz; thus I reduce the four Quantities b, z, x, y, to theſe 1, z, zz, zzz, there are many other ſuch like methods of exterminating unknown Quantities.

CHAP. VI.

22 THE *Terms of an Equation muſt be reduced to the ſimpleſt Expreſſion poſſible, and in ſuch Sort that the unknown Quantity be found only in one of the Members of the Equation.*

WHEN you have reduced the Equation ſo that the unknown Quantity is found only in one of the Members thereof, *that is*, on one ſide of the Sign of Equality, and there are only known Quantities on the other Side; the unknown Quantity is then found exactly equal to known Quantities, and the Queſtion is entirely anſwered. If $x=a+b$, we can't but know the value of x. Now in order to do this, you muſt tranſpoſe that which is known on one Side, and free it from that which is unknown. The firſt thing ſhould be to reduce the Equations to the moſt ſimple Expreſſions. Which is done either by Addition, Subtraction,

---

* *B.* 3. *n.* 60.

† *B.* 3. *n.* 70.

‡ *B.* 3. *n.* 87.

traction, Multiplication, Diviſion, or in ſhort, by the Extraction of Roots. The fundamental Principle of all theſe is that *if equal Quantities are added to, or ſubtracted from, both Members of the Equation, the ſame Equality will continue,* as it will alſo if both Members are multiplied, or divided by the ſame Quantity, as was proved. * Seeing that two equal powers have equal Ratio's, it is plain that in taking the Roots of both Members, the ſame Equality will ſubſiſt.

IF $x-5=15$, add 5 to each Side, and there will ariſe this ſimpler Equation $x=20$. If on the contrary $x+5=20$, by ſubtracting 5 from each ſide, it will be $x=15$. If $x-a=b-a$, adding a to each part, it becomes $x=b$. So likewiſe if $x+a=b+a$, ſubtracting a from each part, it becomes $x=b$. In this Equation $\frac{x}{3}=b$, each Member multiplied by 3, it is $x=3b$, as on the Contrary, if it had been $bx=3b$, each Member divided by b, it would be $x=3$. If $xx=25$, extracting the Square Root, it becomes $x=5$. On the other hand, if it had been $\sqrt{x}=5$, *that is,* if the ſquare Root of x is equal to 5, by involving both members to one and the ſame Power, or by taking their Square, there is this Equation $x=25$. If it had been $xx=ab$, it might be put $x=\sqrt{ab}$. 23

THAT *the Square of the Hypothenuſe is equal to the Squares of both the other Sides,* is a Theorem which ſhews how to reduce an Equation to a ſimpler Expreſſion: For having for Example, this Expreſſion $aa-bb$, and it be required to find a Square equal to $aa-bb$, take the Half of a, and from the Middle or with the Extent of that half, as Radius, deſcribe a Circle; or having taken a Chord equal to b, a Line being drawn from its End to the other End of a, which is the Diameter of the Circle, it will form a right angled Triangle whoſe Hypothenuſe is a, and one of its Sides b: The third Side which is known I call c; then $aa=cc+bb$. Taking away from each Part bb, it becomes $aa-bb=cc$; hence cc may be put in the Room of $aa-bb$, and conſequently we have here a ſimpler Expreſſion. So if I had $cc+bb$. Join c and b together ſo as to make a right Angle, and compleat the Triangle, the third Side is the contained Hypothenuſe: this Hypothenuſe being called a, it is then evident that $aa=cc+bb$, therefore aa, which is a ſimpler Expreſſion, may be Subſtituted inſtead of $cc+bb$. 24

To *find a Square equal to a given Plane.* Suppoſe the two Sides or Roots of the Plane are b and d. Let c be a mean Proportional between them; then $bd=cc$. Thus may the Plane bd 25

---

* *B.* 3. *n.* 54 & 55.

be tranſformed into the Square cc, *that is*, one may be put for
the other.

A Square being given, another may be found bear-
26 ing any required Proportion thereto, either the Half, Third,
Fourth, or Fifth, &c. Let the Square be aa, to find another e-
qual to the fourth Part thereof. Take a Line at venture, upon
which mark five equal Parts with the ſame Extent of the Com-
paſſes. Let this Line be BC, whoſe fifth Part is BD; hence
DC is equal to four of theſe Parts. From A the Middle of BC,
and with the Extent of AB or AC, draw a Circle; and upon D
erect the Perpendicular DE, which terminates at the Circum-
ference of the Circle. Then ∺ BD. DE. DC. * If BD is
called b, then DC$=4b$. And if DE$=x$, then ∺ b. x. $4b$;
therefore $xx=4bb$. Upon C raiſe a Perpendicular CF equal to
a the Root or Side of the Square aa, the Square of whoſe
fourth Part is required. Draw FG parallel to BD; and thro'
the Point G, where it interſects the Radius produced *ad. inf.* let
fall a Perpendicular upon BC produced infinitely on each Side.
The Perpendicular GH is equal to CF, and conſequently to a. †
Draw a Circle with the Extent of AG, whoſe Diameter is IL.
It is evident that IH is the ſame Part of IL as BD is of BC; con-
ſequently a fifth Part. Hence if IH be called m, the Remain-
der of the Diameter HL is $4m$. Now ∺ IH. HG. HL, or ∺
m. a. $4m$: ‡ Then $aa=4mm$. § The Line I H is conſe-
quently the Side or Root of a Square, equal to the fourth
Part of the Square a. This Example ſhews how to find a
27 Square that ſhall be ſuch a Part of a given Square as is re-
quired, which is of great uſe for reducing an Equation to a
ſimple Expreſſion. There are innumerable Ways of reducing
28 compound Expreſſions to others more Simple and Elegant: Af-
ter which there is nothing more to do but to free the Equa-
tion from certain ſuperfluous Quantities, which may be done by
adding that Equation to another, which contains the ſame Quan-
tities with a contrary Sign, according to this principle, *that more
and leſs the ſame Quantity, is equal to nothing*. Thus, if $x=d-b$
and $z=d+b$; in order to cancel b, add the two Equations into
one $x+z=2d$, in which b is expelled. It is eaſy to tranſpoſe a
Quantity from one of the Members of an Equation to the other;
for in this $x-b=d$; adding b to each Side, it will be $x=d+b$, in
which b is moved to the ſecond Member. Altho' the Roots of an
29 Equation be unknown, *that is*, the Quantities of whoſe pro-
duct the Equation is compoſed, yet it may be encreaſed or leſſen-
ed as is deemed neceſſary. Thus if $xx=xd+bb$ whoſe Roots
are x, $x+d$, b, and that it be required to encreaſe x by 3. Take
the Quantity y and ſuppoſe it equal to $x+3$. Then $y-3=x$, then in
every

---

* *B.* 4. *n.* 29. † *B.* 1. *n.* 64. ‡ *B.* 4. *n.* 29.
§ *B.* 3. *n.* 57.

every place where x is, put $y-3$; inſtead of the Square or Cube &c. of x, put the Square or Cube &c. of $y-3$; after which the Equation $xx=xd+bb$, is transformed into this, $yy-6y+9=yd-3d+bb$, which is the ſame. In augmenting or diminiſhing an Equation, it muſt be ſo managed as to have contrary Signs; and that the Quantities may therefore be cancelled which the Equation wants to be freed of.

CHAP. VII.

EQUATIONS *are of one or more Dimenſions or Degrees; and by thoſe Degrees it is that Problems are diſtinguiſhed.* 30

THE unknown Quantities are involved, or raiſed by Degrees, according to the Nature of the Problem. When the unknown Quantity which all the reſt are reduced to, is not multiplied, as in this Equation $x=d+c$, the Equation is ſaid to be ſimple or of one Dimenſion. We have ſhewn that when there are ſeveral unknown Quantities, they are to be all reduced to only one. If for Example there was x and z, and their difference was known to be b; if x be the leſſer, then $z-b=x$; hence $x+b$ may be put inſtead of z. Conſequently; if by the propoſing of the Queſtion, it is known that the Rectangle of x and z, or of x and $x+b$, is equal to the Square of c, there will be this Equation $xx+xb=cc$, which by taking away xb from each ſide, is reduced to this $xx=cc-xb$, in which x is raiſed to the ſecond Degree. If the Queſtion had contained three unknown Quantities, and that they had been multiplied one by the other; x, which all the reſt would be reduced to, would be raiſed to the third Degree. Hence it appears that Equations are of one or more Dimenſions, or of ſeveral Degrees. I ſhall ſay no more here of Equations which are of above two Degrees, becauſe they cannot be ſolved with the Ruler and Compaſs, *that is*, by making uſe of only the Circle and right Line.

EQUATIONS of two Dimenſions are reduced to theſe three forms, 1. $xx=aa-xd$. 2. $xx=aa+xd$. 3. $xx=xd-aa$. For 31 if $xx=ab-d$, as ab is a known Quantity, by taking $ab=cc$, * $xx=cc-xd$. If $xx=xd+c$. I can find a Square bb equal to the Quantity c, ſo that $xx=bb+xd$. Find a mean Proportional between Unit, and c, which being called b, then is $bb=1c$, or $bb=c$. Carefully endeavour to compare the unknown Quantities 32 ſo as may moſt tend to the Formation of the ſimpleſt Equations. By finding and reducing Equations we obtain the Solutions of Problems; now according as theſe Equations are reduced, Problems are diſtinguiſhed. When the Equation is ſo reduced,

---

* Sup. n. 25.

as that it hath only one dimenſion, as if $x=b$, or $x=\frac{ab}{c}$, the E-
quation is of the firſt Degree, and the Problem is a ſimple one.
When the unknown Quantity which the Equation is reduced to,
hath two Dimenſions as $xx=aa+bb$, which is an Equation of
33 the ſecond Degree, the *Problem* is called a *Plane*. When the
Equation contains the unknown Quantity raiſed to the third, or
fourth Degree, as $x^3=aab$, or $x^4=acb$, which are Equations of
the third, and fourth Degree, it is called a *Solid* Problem. When
the unknown Quantity is raiſed higher than the fourth Power,
the Problem was by the Ancients called *Lineal*; but it is more
natural to call it by the Degree which the unknown Quantity
is raiſed to.

CHAP. VIII.

34 OF *the Geometrical Conſtruction or Effection of Equations.* that is, *of the Method of expreſſing by Lines the Quantities concerned therein.*

THE Method of expreſſing by Lines the Quantities belonging to an Equation, is called the Geometrical Conſtruction or Effection of an Equation. The Conſtruction of Equations of one Degree is very eaſy, there being nothing required but to make a Line equal to a known Quantity. Thus if $x=a$, and that a be equal to a foot, it is eaſy with the Ruler and compaſs to draw a Line equal thereto. When the known Member of an Equation hath ſeveral Quantities as $x=a+b$, it is alſo eaſy, for the Equation may be reduced to a ſimpler Expreſſion, by taking a Quantity equal to both the Quantities. Two different Lines may be taken and joined together. Sometimes inſtead of the value of a Quantity we ſeek its quotient when divided by other Quantities; for Example $x=\frac{ab}{c}$: To expreſs this Geometrically, *that is*, to conſtruct or make a Figure which expreſſes this Equation. Obſerve that theſe Letters denote Quantities proportional one to another; for if $c.\ a::b.\ x$, the ſecond Term a being multiplied by the Third b, and their product ab, divided by the Firſt c, the Quotient of the Diviſion $\frac{ab}{c}$ is x the Fourth proportional; hence $x=\frac{ab}{c}$. To expreſs this Geometrically, having taken AE equal to c, and AE equal to b, upon B, raiſe BC$=a$, and from A draw a Line thro' C; then from E, draw a Parallel to BC or to a, which I call x. Theſe four Lines

AB. BC∷AE. x. are proportional; hence $\frac{BC\times AE}{AB}$, or $\frac{ab}{c}$
$=x$. So likewiſe to expreſs geometrically $\frac{aa+ab}{c+d}=x$, as 35
theſe known Quantities may be reduced to this Proportion $c+d$.
$a+b$∷a. $\frac{aa+ab}{c+d}$ or x, the ſame figure expreſſes this Equation;
three lines being given a fourth proportional may be found,
which is the value of x.

When the known Quantities of an Equation have a radical 36
Sign, they may be likewiſe expreſſed geometrically. If for
example, $x=\sqrt{ab}$, by drawing a line AB equal to a, and pro-
ducing it even to C, ſo that BC$=b$, and from the middle of
AC, as a center, deſcribe a circle, then is the perpendicular
BE the Value of x; for ∺ AB. BE. BC, or ∺ a. x. b; hence
$ab=xx$: Therefore $\sqrt{ab}=x$. This Equation $x=\frac{bb}{a-b}$ may be al- 37
ſo conſtructed and expreſsed thus. AD being taken equal to a,
and AB to $a-b$, hence BD being equal to b, I raiſe upon B a Per-
pendicular equal to BD or b, draw a right line from A to E; and
upon AE at the point E draw a perpendicular, *viz.* EC, which
cuts the produced Part of AD: then is the Angle CEA a right
one; having therefore deſcribed a circle thro' the three Points
A, E, C, it is evident that ∺ AB. BE. BC, or that ∺ $a-b$. b. x.
put x for the value of BC, then bb, divided by $a-b$ is equal to
x; * hence $\frac{bb}{a-b}=x$. Equations of two Degrees are alſo eaſily 38
conſtructed; and are, as we have ſhewn, reducible to one of
theſe three Forms $xx=ax+bb$, or $xx=ax-bb$, or $xx=bb-ax$.
1. For the Conſtruction of $xx=ax+bb$, or $xx=bb-ax$; upon
a right line CD, which I ſuppoſe equal to b, I erect a perpen-
dicular CE which I ſuppoſe equal to the half of a, and with the
extent of that half, as Radius, I deſcribe a Circle, and thro' its
Center E, draw the Line DE. Since that CE is the half of a,
then AB$=a$. Let BD$=y$, and AD$=x$. then $a+y=x$; there-
fore ‖ $ax+xy=xx$. Now ſeeing that CD or b is a Tangent,
then † ∺ x. b. y; therefore ‡ $xy=bb$. Hence ſubſtituting bb in
this Equation $xx=ax+xy$ in the Room of xy, which is the ſame,
there will ariſe this Equation $xx=ax+bb$, which is the con-
ſtruction required. 2. The conſtruction of this Equation $xx=$ 39
$bb-xa$, is the ſame. CD (*ſame Figure*) is ſuppoſed equal to b,
and CE equal to the Half of a; but 'tis ſuppoſed that BD is
equal to x. Then ∺ AD(or $a+x$.)CD (or b.) BD (or x:) then
ax

---

* *B.* 3. *n.* 60.

‖ *B.* 3. *n.* 18.

† *B.* 4. *n.* 33.

‡ *B.* 3. *n.* 57.

$ax+xx=bb$. By taking away ax from each part, there will remain $xx=bb-ax$, which is the Equation required to be conſtructed. 3. This Equation $xx=ax-bb$, is conſtructed thus. 40 Let BA be equal to a, with the extent of BC, the half of AB, deſcribe a Circle; upon B, erect the Perpendicular BD equal to b; thro' D draw the Line DG Parallel to AB. If this parallel don't croſs the Circle, which will be the Caſe if b is equal or greater than the Radius of the Circle, which is equal to the Half of a, 'tis a Proof that xx is not equal to $ax-bb$, ſince that b is too great with reſpect to x. Let then x be leſſer than BC, hence DG interſects the Circle in E, conſequently EF which I ſuppoſe perpendicular, is equal to BD. § Suppoſe AF$=x$, and FB$=y$. It is evident * that ∺ AF. (or x.) FE. (or b.) FB. (or y.) hence ∺ x. b. y: therefore $xy=bb$. Now AF (or x) more FB (or y,) hath been ſuppoſed equal to a; therefore $x+y=a$. This Equation multiplied by x, produces $xx+xy=ax$: Subſtituting bb, in the room of its Equal, xy, it is $xx+bb=ax$; and ſubtracting bb from each Side, it will be $xx=ax-bb$, thus is this Equation conſtructed. 41 By means of this Conſtruction the Value of the unknown Quantity is found by a Line; which is found yet more readily by reducing the three preceeding Forms, into a Progreſſion of three Terms. 1. This Equation $xx=ax+bb$ may be reduced to this progreſſion ∺ $x-a$. b. x; for $xx-ax=bb$; § adding ax to each Part, it is $xx=ax+bb$. 42 2. This Equation $xx=ax-bb$, is reducible to this progreſſion ∺ x. b. $a-x$. for $ax-xx=bb$, † adding xx to each ſide, it is $ax=bb+xx$; taking away bb from each Side, it is $ax-bb=xx$. 43 3. This Equation $xx=bb-ax$, is reducible to this progreſſion ∺ $x+a$. b. x; for $xx+ax=bb$: ‡ taking away ax from each ſide, it is $xx=bb-ax$. The mean Term b is known in each of theſe three Progreſſions, and in the firſt and third, the Difference of the Extremes is known, which in the firſt is $-a$, and in the third $+a$. In the ſecond, the Sum of the Extremes is known. With theſe all the other terms of the progreſſions may be found, as will appear by the three following Problems.

44 PROBLEM I. LET *there be a Progreſſion of three Lines* ∺ *z. b. x. the mean Term is b, and a is equal to the Sum of the Extremes x and z. required to find the Extremes.*

Take AB equal to a, the Sum of the Extremes. From C, the Middle of AB as a Center, and with the Extent of AC or CB, deſcribe a Circle. Upon B raiſe perpendicularly BD equal to the mean b; and thro' the Top D draw DG Parallel to AB, and from E where DG cuts the Circle, let fall a Perpendicular FE, upon AB, which is equal to BD, hence EF$=b$; then ſeeing that ∺ AF. EF. FB. the Extremes are AF and FB; therefore ſore if x be the greater Extreme, FB which is the leſſer Extreme, is equal to x.

---

* *B.* 4. *n.* 29. § *B.* 3. *n.* 58. † *B.* 3. *n.* 57. ‡ *B.* 3. *n.* 57.

PROB. 2. LET *there be this Progreſſion of three Lines* $\div\div$
$x+d.\ b.\ x.$ *or* $\div\div\ x-d.\ b.\ x.$ *the mean* b *is known, and the* 45
Difference of the Extremes $x+d$ *and* x. *To find the Value of* x.

DRAW AB equal to d; and upon B raiſe Perpendicularly BD equal to b; then from C the half of AB, and with the Extent of CD, deſcribe a Circle. Prolong AB at each end even to the Circumference of the Circle, after which AE, or BF$=x$. For $\div\div\ x+d.\ b.\ x.$ If the Progreſſion were $\div\div\ x-d.\ b.\ x.$ the ſame thing muſt be done. But in the Caſe where $x=$EB, the Leſſer Term is BF equal to EB–AB, or to $x-d$. It is evident that the difference between $x-d$ and x, is d. When the Sign is $+$, the Quantity x, is EA, to which is added the difference d, which gives the greater Extreme. When the Sign is $-$, d is ſubtracted from EB or FA which is the Value of x.

PROB. 3. LET *there be this Progreſſion of three Lines* $\div\div\ d+x.$ 46
$b.\ d,$ *the mean Term* b, *being known, to find the Difference of the Extremes* $d+x$ *and* d, *which is* x.

UPON B, the Extremity of AB, equal to d, raiſe Perpendicularly BC, equal to b; thro' A and C draw a right Line, upon which at the Point C draw a Perpendicular which is CD; therefore the Angle ACD is a right one. Produce AB even till it cut CD; the Line BD–AB is equal to x; for having divided AD into two equal parts at the Point K, and with the Extent of AK or KD deſcrib'd a Circle, it will paſs thro' C; hence $\div\div$. AB. *(or d.)* BC. *(or b.)* BD: Wherefore BD$=d+x$, which is the third Term: having then ſubtracted from BD the Line DE equal to AB, the Remainder is equal to x; whoſe value was required.

CHAP. IX.

OF *the geometrical Conſtruction of a* Locus *or* Place.

A geometrical Place is the Line by which a Problem is 47
conſtructed, and as there are various Sorts of Lines, ſo there are different ſorts of *Places*. When only a right Line is required, it is called a Place for a right Line. If a Circle muſt be uſed, then it is a Place for a Circle, &c. Indeterminate Problems are alſo of Place. Let us begin by a Place which is a Line. The Line Line LL is one, if there can be drawn from all its points the parallel Lines LN, LN, meeting a right Line AN; and having taken upon the Line AN a Point A at pleaſure, each Line LN hath one and the ſame Proportion to the Part AN, which they contain. For Example, if LL is a right Line, and it

it meets the right Line AN in A, it is evident that every right Line LN hath the ſame Proportion to each part of AN, which may be expreſſed by this Equation $y = \frac{bx}{a}$. If the Proportion be ſuppoſed to be as that of a to b, and that the Indeterminate Lines are LL$=x$, and NN$=y$. For ſince that $a.\ b :: x.\ y$; then the Product of b by x, which is bx divided by a, is the Value of y; hence $y = \frac{bx}{a}$. This Equation ſhews that the Place is a Line; for 'tis only a Line that can have this Property, *viz. That all Lines drawn from AL upon AN, are to AN as a is to b.* Hence it appears that the Problem which required to find LL is a Place; and the Problem admitting of different Solutions, is therefore indeterminate, and the Place can be only a right Line; for there is only that which place LL poſſeſſes, which hath the Properties of the right Line,

49 A Problem is a place of a Circle, and indeterminate, when it is propoſed to find a Line having its Square equal to a Plane; for then known Quantities muſt neceſſarily be ſuppoſed, as for Example, that the Sides of the Plane are a and b; that AB$=a$, and BD$=b$. AD$=a+b$, and AC$=$CD. Upon C as a Center deſcribe a Circle, and upon B draw the Perpendicular BE, then $\div$ AB. BE. BD; then $ab = \overline{BE}^2$; hence if BE$=x$, then $ab = xx$, or $\frac{ab}{x} = x$. The Problem is indeterminate: For whatever Ratio I ſuppoſe to be between the parts of AD, the Square of the Line which will fall Perpendicularly between the two Points A and D, will always have its ſquare equal to the Plane of the parts of AD, *that is*, that $ab = xx$, or $\frac{ab}{x} = x$. The Problem may therefore have innumerable Solutions. You ſee that to find an Equation we were obliged to ſuppoſe the known Lines AB and BD, otherwiſe it could not be done. There is only the Circle which hath always this Equation in all its parts.

C H A P. X.

50 SOLID *Problems are not ſolvable by the Knowledge of theſe Elements, for there are only ſimple Equations, or thoſe of two Degrees, which can be expreſſed Geometrically, with the Rule and Compaſs.*

IT has been ſhewn how to ſolve Equations of one or two Degrees by the Rule and Compaſſes, *that is*, by drawing right Lines and deſcribing Circles, but this will not do, when they have more Degrees. See here an Example, in the triſection of an Angle which is a noted Problem. An Arc of a Circle being given, it is propoſed to divide it into three equal parts, and conſequently to find a third Part of the propoſed Angle, whoſe meaſure is the third Part of that Arc. The Chord BE of the Arc BCDE is known, it is propoſed to divide this Arc into three equal parts, ſuppoſe the thing as done, that $BC=CD=DE$; draw CF parallel to DH; hence $CF=DH$: and ſince that $CG=DH$, therefore $CF=CG$; hence the Triangle FCG is Iſoceles. The Angles BCG and BGC are equal; the Meaſure of BCG is half the Arc BK, and that of BGC is the half of KE, more the half of BC or DE equal to BC. * Now BK is equal to KD; then theſe two Angles having the ſame Meaſure are equal: Therefore the Triangle CBG is Iſoceles. It hath one Angle common with FCG; theſe two Iſoceles are therefore Similar. BAC is alſo Iſoceles, and has an Angle common with CBG, *viz.* BCG; theſe three Triangles are therefore Similar. † Theſe three Triangles BAC, CBG, FCG being Similar, then ∺ AB. BC. CG. GF, ‡ then if $AB=1$, and $BC=z$, according to what was ſaid above, § ∺ $1.\ z.\ zz.\ zzz$; then $FG=zzz$. The known Line EB I call b. Now $CD=FH=BC=BG=HE$; then $BG+FG+GH+HE$ is triple of BC; therefore there is wanting the Value of FG to make BE or b the triple of BC. Now $FG=zzz$, therefore $b+zzz=3z$, or $zzz=3z-b$. This is as far as we are able to proceed in this Problem. But we cannot find the unknown Quantity by help only of theſe Elements, it being only known that its Cube zzz is equal to three times itſelf, *that is*, to $3z$, the known Quantity b being deducted. This Problem is eaſily ſolved Mechanically. Let the Arc BC, be the Meaſure of the Angle BAC, which is to be divided into three equal Parts. Produce the Diameter CF towards E, apply a Ruler upon B, and upon the produced Part of CF, find in the Circle ſuch a Point D, that the Line AD may be equal to DE; which done, I ſay that the Arc DF is the third Part of BC, or that DAF is the Third of BAC; which I demonſtrate thus, ADE and DAB are Iſoceles; then $DBA=BDA$, and $DAE=DEA$: the exterior Angle BDA, is equal to the two Interior ones DEA and DAE; therefore DBA is equal to them both, and conſequently is double either of them. The exterior Angle BAC is alſo equal to the two Interior ones DEA *(or its equal DAF)* and to DBA, therefore it is triple of DAF, the half of DBA; *Q. E. D.*

---

* *B.* 2. *n.* 39 & 52.

† *B.* 2. *n.* 83 & *B.* 4. *n.* 8.

‡ *B.* 4. *n.* 10.

§ *Sup.* *n.* 21.

Several other things concerning Equations might be added; but here is ſufficient, for to make uſe of what could be ſaid more, requires a Knowledge in the Doctrine of Curve Lines.

CHAP. XI.

52 THE Solution of a Problem may be attempted two different ways. The Firſt is only an Application of the former Part of theſe Elements, which diſcovers ſome courſe Particular to the Problem in hand, and which cannot be uſed in any other. The Second, is the Order preſcribed by the Method which we have juſt taught, according to which the Thing ſought is found in a much more excellent Method, as it extends univerſally to every Problem. Here follows an Example of each.

PROBLEM I.

53 BAC *is Iſoceles. It is required to divide the Sides AB, AC, by a Line Parallel to the Baſe, ſo that the Parallel may be equal to the Remainder of the Sides,* that is, *ſuppoſing the Thing done,* that DB=DE.

METHOD I.

SUPPOSE the Thing done, *viz.* That BD=DE, then the Triangle BDE is Iſoceles; hence the Angles DBE, and DEB are equal. Now the Angles CBE and EBD are alſo equal.* Wherefore EBC and EBD are equal; therefore the Line BE divides the Angle DBC equally in two. From whence I know that in an Iſoceles Triangle, ſuch as BAC, if the Angle ABC be divided into two equal Parts by a right Line BE, and there be drawn thro' E a Parallel to BC, it will be equal to DB; therefore by this property of the Iſoceles Triangle, I find the Way to ſolve the propoſed Problem; but this Method is, you ſee, particular and proper to this Problem only.

METHOD 2.

54 SUPPOSING the Thing done, the known Line AB I call a, and d the Baſe BC which is known alſo, and the unknown Quantity AE which is ſought I call x; hence as EC$=a-x$, alſo DE$=a-x$, it is evident that $a.\ d::x.\ a-x$, then $aa-ax=dx$. Add to each Side ax, and it becomes $aa=dx+ax$; ſuppoſe $c=d+a$, hence $cx=dx+ax$; and conſequently putting cx inſtead of $dx+ax$, I have $aa=cx$; then ÷

---

* *B.* 2. *n.* 25.

$c. a. x$: Therefore the Buſineſs is only to find a third Proportional to the two known Lines c and a, which has been taught, *B. 4. n. 23.* This ſecond Analytical Method is univerſal, and not particular to this Problem only.

PROB. 2. THE *Sum of any two Sides of a Triangle ABC being known, to find each Side thereof.* 55

LET $AB=x$, and $AB+BC=a$, and $AB+AC=b$, and $BC+AC=c$; then $BC=a-x$, and $AC=b-x$. Now $AC+BC=c$; then $a-x+b-x$, or $a+b-2x=c$. Then adding $2x$ to each Side, it will be $a+b=c+2x$. Taking away c from each, it is $a+b-c=2x$. Therefore to find the Value of x, join the two Lines a and b together, from their Sum ſubtract c, then is half the Remainder the Value of x.

PROB. 3. ONE *of the Sides of a right angled Triangle is a, and the Other is x, whoſe difference with the Hypothenuſe is b. To find the Value of b.* 56

THE Hypothenuſe is $x+b$; then $aa+xx=xx+2bx+bb$. Taking away xx from each Side; $aa=2bx+bb$; alſo ſubtracting bb, it will be $aa-bb=2bx$; taking $cc=aa-bb$, * it will be $cc=2bx$. Then ∺ $2b.\ c.\ x$. The Buſineſs is therefore only to find a third Proportional to the two known Quantities, $2b$ and c.

PROB. 4. The *Hypothenuſe of right angled Triangle is a, and the Difference of the two Sides b. To find the Sides.* 57

LET the leſſer Side be called x; then the Greater is $x+b$; hence $aa=2xx+2xb+bb$. Subtract bb equally from each Side and $aa-bb=2xx+2xb$, or half aa–half $bb=xx+xb$. In the Place of half aa—half bb, I put the Square cc, which I find to be equal thereto. Then is $cc=xx+bx$, or $cc-bx=xx$; hence ∺ $x+b.\ c.\ x$. Hence may be found the Value of x.

PROB. 5. THE *Hypothenuſe of a right angled Triangle is a, and the Sum of the two Sides b. To find the Sides.* 58

LET one of the Sides be called x; then the other is $a-x$: Hence $aa=bb-2bx+2xx$. I add $2bx+aa=bb+2xx$. I alſo ſubtract bb, and I have $2bx+aa-bb=2xx$. Put dd in the Place of its equal $aa-bb$, then $2bx+dd=2xx$. Divide the whole by 2, in order to which find cc, equal to the Half of dd; † hence $bx+cc=xx$: Then ∺ $x.\ c.\ x-$. Then the Value of x may be eaſily found.

PROB. 6. THE *Area or Superficies of a right angled Parallelogram is aa, its leſſer Side x, and x+b, the greater Side, required to find the two Sides.* 59

THE Product of x by $x+b$ is the Area of the Parallelogram; then $xx+xb=aa$: hence ∺ $x.\ a.\ x+b$. This Problem is conſequently Solved like the preceeding one.

PROB. 7. THE *longeſt Side of a right angled Triangle is x, its difference with the Hypothenuſe is a, which is therefore x+a,* 60

---

* *Sup. n.* 24. † *Sup. n.* 26.

the Difference of the Hypothenuſe with the other Side which is leſſer than that, is b, hence the Side is $x+a-b$. *Required to find the three Sides.*

The Squares of the two Sides x and $x+a-b$ are equal to that of the Hypothenuſe, which is $x+a$, hence $2xx+2ax-2bx-2ba+aa+bb=xx+2ax+aa$. Add $2ba$ to each Side, and it is $2xx+2ax-2bx+aa+bb=xx+2ax+aa+2ba$. Subtract $xx+2ax+aa$ from each part, and it is $xx-2bx+bb=2ba$, ſubtract $+bb$, and it is $xx-2bx=2ba-bb$: Taking d equal to $2b$, and cc equal to $2ba-bb$, it will be $xx-dx=cc$; hence ∺ x. c. $x-d$. Which is eaſily Solved. *

61 Prob. 8. The *Perpendicular drawn from the right Angle of a right angled Triangle upon the Hypothenuſe is b; the Difference between the two Parts or Segments of the Hypothenuſe is a. To find the Segments.*

Let the leſſer Segment be x; then the Greater is $x+a$. Now ∺ $x+a$. b. x. † Then the Problem is ſolved as has been taught. ‡

62 Prob. 9. The *Line AB is divided in one of its Points, as C; it is required to produce it to D, ſo that the Rectangles comprehended under AD and BD may be equal to the Square of CD.*

Suppose the Thing done. Let AC$=a$, and CB$=b$, and BD$=x$. It is required to find the Value of BD, or x. Multiply AD, or $a+b+x$ by BD, *that is,* by x, which makes $ax+bx+xx$, which Product according to the Queſtion is equal to the Product of CD, or of $b+x$ multiplied by itſelf; *that is,* $ax+bx+xx=bb+2bx+xx$. From both Members of this Equation, ſubtract $bx+xx$ and there remains $ax=bb+bx$; tranſpoſe bx to the other Side, in order that the unknown Quantity bb may remain alone, and then $ax-bx=bb$. To reduce this Equation to ſimpler Terms, divide it by $a-b$, then $x=\dfrac{bb}{a-b}$, which Equation is reſolved into this Progreſſion ∺ $a-b$. b. x, the two firſt Terms of which being known, the Third x, which is ſought, is alſo known; therefore to conſtruct the Problem produce the Line AB equal to x.

The *Solution of each Problem affords a new Theorem: For according to what has been proved, the Square of BD, the produced Part of a Line, more CB part of that Line, is equal to the Rectangle made of AD and BD; and the produced Part is the third Term of a Progreſſion, whoſe firſt Term is AC—BC, and the Second BC. The moſt part of Theorems are deduced from Algebra, which is a fruitful Source of truth.*

63 Prob. 10. The *Line AB is divided in C. It is propoſed to divide it again in D; ſo that the Rectangle of AC+DC by CD, may be equal to the Square of DB.*

---

* *Sup. n.* 38 *&* 41. † *B*: 4. *n.* 28.

‡ *Sup. n.* 38. *&* 43.

SUPPOSE the Thing done. Then to find the Value of CD; let $AC=a$, and $CB=b$, and $CD=x$; hence $DB=b-x$. The Rectangle of AD by CD is $ax+xx$, and the Square of DB is $bb-2bx+xx$; then according to the Queſtion $ax+xx=bb-2bx+xx$. Subtract xx from both Sides, and $ax=bb-2bx$. Tranſpoſe $2bx$ to the other Side, in order that the known Quantity may be alone, $ax+2bx=bb$; divide this Equation by $a+2b$, then is $x=\frac{bb}{a+2b}$; turn this Equation into a Proportion $::$ $a+2b.\ b.\ x.$ the two firſt Terms are known, therefore x is known alſo; hence by taking upon CB the Line CD equal to x, there will be done what was required.

PROB. 11. THE *right Line AB is divided at C, the infinite Line BD is Perpendicular upon AB; required to draw from A, a Line AD upon BD, ſo that* $AD=BC+BD$. 64

SUPPOSE the Thing done, and that $AB=a$, and $BC=b$, and $BD=x$, the Line required. According to the Queſtion $AD=BC+BD$, wherefore $AD=b+x$. Now ſince that ABD is a right Angle, the Square of AD or of $b+x$, which is $bb+2bx+xx$, is equal to thoſe of AB, and of BD, which are aa and xx. Hence $bb+2bx+xx=aa+xx$; taking away xx from both Sides, it is $bb+2bx=aa$. In order that the unknown Quantities may be all on one ſide, tranſpoſe bx, and it will be $2bx=aa-bb$, which divided by $2b$, gives $x=\frac{aa-bb}{2b}$, which Equation is therefore reducible into this Proportion, $2b.\ a+b::a-b.\ x$, whoſe three firſt Terms being known, the Fourth x; which was ſought, is alſo known.

PROB. 12. Two *parallel Lines AB and CD, are given by Poſition with CF, as alſo the Points F and E. It is required to draw FD interſecting AB and CD, and produced if needful, ſo that AB be to ED, as AF is to AG.* 65

LET $AF=a$, $CF=b$, $CE=c$, $AG=d$, and $AB=x$. I ſuppoſe the Thing done. Therefore according to the Hypotheſis $a.\ d::x.$ ED. Then x multiplied by d, and the Product which is dx divided by a, the Quotient $\frac{dx}{a}=$ ED. The two Triangles AFB and FCD are Similar; $a.\ x::b.$ CD. Now $CD=CE+ED$, and therefore $CD=c+\frac{dx}{a}$. The Product of the Extremes a and $c+\frac{dx}{a}$, which is $ac+dx$, is equal to that of the Means: Then $ac+dx=bx$. Tranſpoſe $+dx$, and it is ac

$=$

$=bx-dx$. This Equation divided by $b-d$, gives $x=\frac{ac}{b-d}$, which Equation is reducible to this Proportion $b-d.\ c::a.\ x$, whoſe three firſt Terms are known.

66 PROB. 13. To *meaſure an inacceſſible Height AB, and its diſtance AC, by the Help of two Sticks CD and EF.*

SUPPOSE $AC=x$, $AB=y$, $CE=b$, $GD=a$, and $FH=c$; then ſeeing that the Triangles BAF and GDF are Similar, AB. GD ::FA. FG; but FA. FG::FI. FK, upon account of the Parallels GD, and BA; then AB. GD::FI. FK, or their equals AE and CE, hence $y.\ a::x+b.\ b$, conſequently $by=ax+ab$. The two Triangles BHF and BGD are Similar; then as BD is to BF, or IK to IF, or AC, to AE, ſo likewiſe GD is to HF; hence AC. AE::GD. FH, or $x.\ x+b::a.\ c$, therefore $xc=ax+ab$; taking ax from both Sides, there remains $xc-ax=ab$: dividing each Member by $c-a$, it will be $x=\frac{ab}{c-a}$; hence $c-.\ a::b.\ x$. The three firſt Terms of this Proportion are known; therefore the Fourth which is required is alſo known. Seeing it appeared above that $by=ax+ab$; and $xc=ax+ab$, it follows that $by=cx$, it being equal to one and the ſame Quantity $ax+ab$, wherefore $b.\ c::x.\ y$. Now x is known already: Therefore the three firſt Terms being known, the fourth Term y which is required, is known.

67 PROB. 14. Two *Merchants put into Company* 1200 *Pounds, and they gained* 3400 *Pounds; the Firſt took* 700 *Pounds, for both his Gain and Sum put in for two Months; the Second took* 3900 *Pounds, for his Gain and Sum put in for five Months. It is demanded what each Gained, and alſo the particular Sum put in by each?*

THE Sum put in by both is $1200=a$. Let the Sum put in by the Firſt be called x; hence that of the Second is $a-x$. The Stock and Gain of the Firſt is $700=b$: Then $b-x$ is the Gain of the Firſt. The Stock and Gain of the Second is $3900=c$; therefore $c-a+x$ is the Gain of the Second. Now as the Stock of the Firſt multiplied by its time, is to its Gain; ſo the Stock of the Second multiplied by its time, is to its Gain; *that is*, $2x.\ b-x::5a-5x.\ c-a+x$. The Product of the Extremes is equal to that of the Means; then $2xc-2ax+2xx=5ab-5ax-5bx+5xx$; and $2ax$ being added, and $2xx$ at the ſame Time ſubtracted from both Sides of the Equation, it will be $2xc=5ab-3ax-5bx+3xx$; add $3ax+5bx$ to both Sides, and it gives $2cx+3ax+5bx=5ab+3xx$. Suppoſe $d=2c+3a+5b$; hence $dx=5ab+3xx$; alſo take $f=5ab$, which ſubtract from each Side, and then $dx-f=3xx$. Then to reduce this Equation to a form which gives the Solution of the Queſtion, ſuppoſe $d=3g$, and $f=3b$; hence $gx-b=xx$, find a Square equal to b, which I call ll, wherefore $gx-ll=xx$; which Equation is reducible to this Progreſſion $\div\div\ g-x.\ l.\ x$, for $gx-xx=ll$, and conſequently $gx=ll+xx$, and $gx-ll=xx$, the Extremes of this Progreſſion are $g-x$ and x, whoſe Sum is g. Their difference will be found if upon AB$=g$ be deſcribed a Circle AGB, and there be raiſed a Perpendicular BD$=l$, and DG be drawn Parallel to AB, and from the Point E, be drawn the Perpendicular EF, it will cut AB at the Points required. The unknown Quantity which is ſought is x. Now as we in this Problem ſeek the Value of x in numbers, there muſt from the Square of CE *(or CB, the half of AB, the Sum of the known Extremes,)* be ſubtracted the Square FE$=ll$, which is alſo known, and there will remain the Square of CF, the Half of the Difference of the Extremes, whoſe Root added to CB will give the greater Term, and being ſubtracted it gives the Leſſer $x=300$, the Stock of the Firſt; after which all the Reſt is eaſy.

ADVERTISEMENT.

THE *two following Problems ſhew that an extenſive uſe may be made of this Method of ſolving Queſtions, which tho' ſeemingly as not appertaining to Geometry, yet their Solution depends upon our Knowledge herein.*

PROB. 15. LET *ABCD be a Billiard Table. Required to ſtrike the Ball H after two Rebounds, with the Ball K.* 68

THE Angles of Incidence and Reflexion are equal. According to this Principle, it is ſuppoſed that if K is puſhed to the Point I it ought to reflect to L, and from L to H. The Point I is then required. Having drawn thro' K the Line EF, parallel to the Side CD; and thro' the other Point H, the Line GH Parallel to BD. Let KF$=a$, FB$=b$, BG$=d$, GH$=f$, and FI $=x$: Then BI$=b-x$. The Triangles KFI and IBL being Similar, $x.\ a::b-x.$ BL. Then BL$=\dfrac{ab-ax}{x}$, and by contracting this Expreſſion BL $=\dfrac{ab}{x}-a$. Now GB$-$BL$=$ GL: Then taking away $\dfrac{ab}{x}-a$ from d, it will be GL$=d+d-\dfrac{ab}{x}$. The Triangles KFI and LGH are ſimilar: Therefore $a.\ x::d+a-\dfrac{ab}{x},\ f$: Therefore $af=dx+ax-ab$: Then adding ab to each Side and dividing by $d+a$, it is $\dfrac{af+ab}{d+a}=x$; and reſolving this Equation into a Proportion, it will be $d+a. f+b:: d. x$; *that is,* FI is a fourth Proportional to theſe three Lines BG+KF, GH+FB, and KF; which are known Lines. Wherefore FI is the Line ſought, therefore the Point is I. Therefore if the Ball K is to touch the Ball H after two Rebounds, K muſt be ſtruck ſo that it go to I, from whence it will rebound to L and from L to H; which is the Thing propoſed.

69 **Prob.** 16. Two *Points or two Places being given, with a Plane at a Diſtance from thoſe two Points, required to find upon this Plane, a Point which muſt be paſſed thro' in order to go the ſhorteſt Way from one Place to the other.* Or *a luminous Object being given with a* Mirrour, *and the Eye placed out of the* Mirrour. *To find the Point of Reflection.*

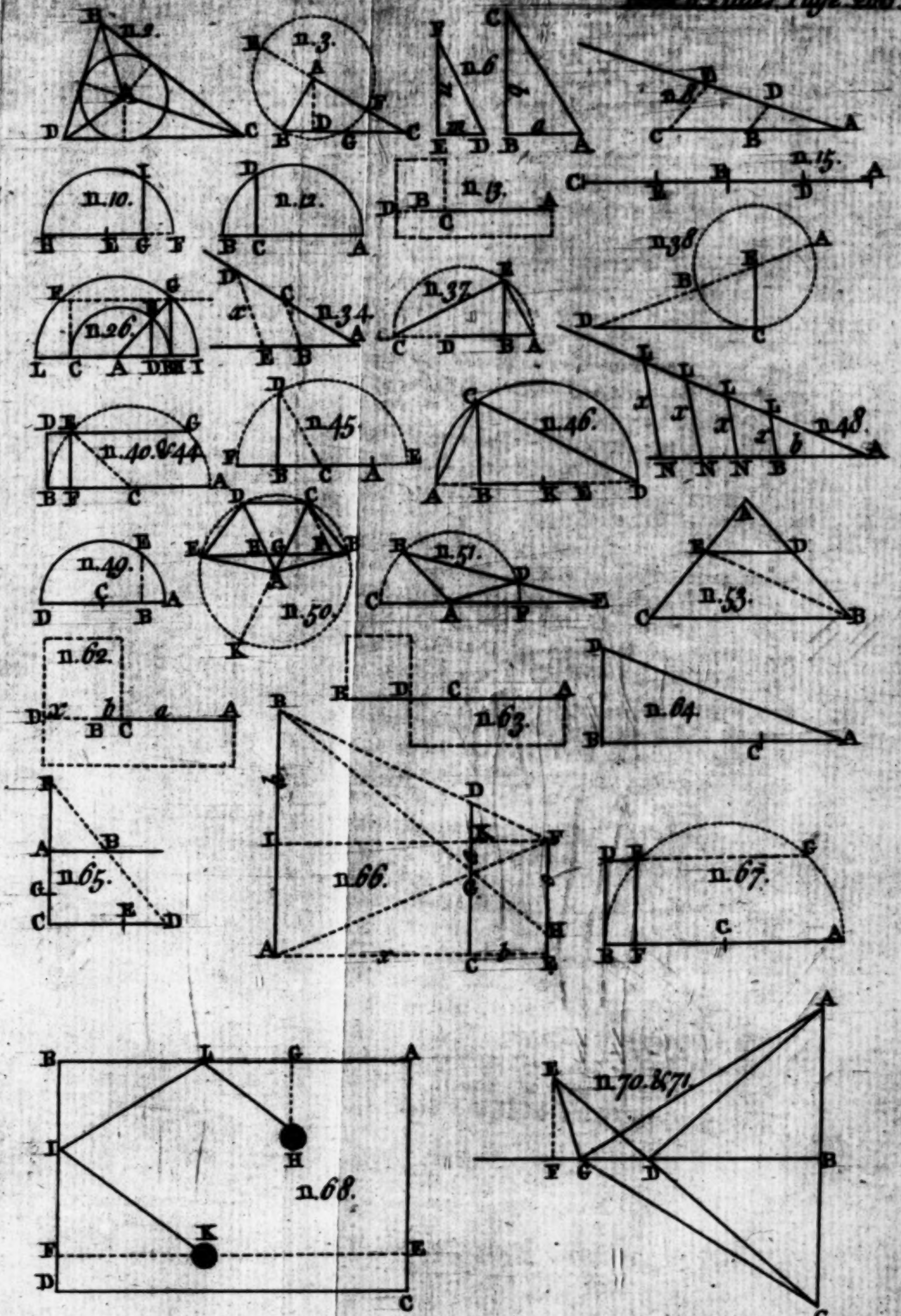
Let the given Points be A and E; and let alſo the Plane be BF. Required to find upon that Plane the Point D, ſuch that the Line AD more DE, may be ſhorter than any other of the Lines drawn from that Point to the Plane, for Example, ſhorter than AG+GE, &c.

70 *FIRST.* Let there be drawn the Perpendicular AB produced even to C, ſo that AB=BC. And from the Point E, let there be drawn the Line CE; I ſay that the Point Sought is D: For the two Triangles ABD and DCB are equal, as are ABG and CBG: Then AD+DE=CD+DE, and AG+GE=CG+GE. But CD+DE is a right Line: It is therefore ſhorter than CG+GE. If the Point G were between B and D, the Demonſtration would be the ſame. Obſerve that the two Triangles ABD and CBD being equal, they are Similar: Therefore the Angles ADB and BDC are equal; but EDF is equal to BDC: Then EDF is equal to ADB; hence as the Angle of reflection is equal to that of Incidence, if the Object is at E, and the Mirrour be BF, the Point of Reflection is D, and the Light willbe reflected at A. It is therefore conveyed by the ſhorteſt Way.

71 *SECONDLY.* The Angle of Incidence EDF is ſuppoſed equal to that of Reflexion ADB. The Buſineſs is to find again the Point D, where the Light will fall reflected from the Object E to the Eye A, proceeding by the ſhorteſt Way. The known Quantities are $AB=a$, $EF=b$, and $BF=d$. The unknown is $BD=x$. Seeing that the Angles of Incidence and Reflection are ſuppoſed equal, then the two right angled Triangles ABD and EFD are Similar; then $a+b. a::d. x$.

Therefore $\frac{ad}{a+b}=x$: Hence the Value of BD is known.

The *End of the Elements of Geometry.*


n.1.
n.3.
n.6
n.8.
n.10.
n.12.
n.13.
n.15.
n.26.
n.34.
n.37.
n.38.
n.40.&44.
n.45.
n.46.
n.48.
n.49.
n.50.
n.51.
n.53.
n.62.
n.63.
n.64.
n.65.
n.66.
n.67.
n.68.
n.70.&71.
A
B
C
D
E
F
G
H
I
K
L
N
a
b
x
m
n

AN
INTRODUCTION
TO
CONIC SECTIONS.

CHAPTER I.

OF *the curve Lines which repreſent the different Sections of a Cone. Their Names, and the eaſieſt Method for finding their principal Properties.*

In theſe geometrical Elements I have treated only of the right Line and the Circle; but as there are other Lines called curve Lines, which are now-a-days chiefly the Object of the moſt ſublime Geometry, upon account of the wonderful and ſingular Properties which they diſcover, it will be proper to ſhew their *Geneſis*, and ſome of their principal Properties. I ſhall treat only of the Firſt and moſt known, which are thoſe obſerved in cutting the Cone different ways, from thence, called *Conic Sections*. The Cone and its ſeveral Sections are ſhewn in P. B. Plate 8.

Let there be conceived a right angled Cone. 1. If it be cut by a Plane paſſing thro' its Axis, it is evident that the Section will be a Triangle, whoſe Sides are thoſe of the Cone, and its Baſe the Diameter of the Circle which is the Baſe of the Cone. *In what follows I ſuppoſe the Plane which divides the Cone, to be Perpendicular upon the Plane of the Triangle.* 2. If the Section be Parallel to the Baſe of the Cone, it is evident the Section is a Circle. 3. If it be not Parallel to the Baſe, but interſects the Cone's Axis, and both its Sides, (*I mean the two Sides of the Triangle, which is the Section of the Cone by a Plane paſſing thro' its Axis*) this Section, or the Circuit of this Section will repreſent a curve Line called an *Ellipſis* or *Oval*. Such is the Figure *x*, formed in the Cone A by cutting it in the Direction of the Line M. *See this Section PB. Plate* 8. *fig.* 30. 4. If the cutting Plane divides only one of the Sides of the ſaid Triangle, and that the Section be Parallel to the other Side, this Section or the Circuit thereof is a different Curve, called a *Parabola*. Such is the curve Figure Y, formed in the Cone B, by the Plane which cuts it, according to the Line N. *See P. B. P.* 8. *fig*. 32. If the Section be ſuch that if prolonged it croſs the other Side of the ſaid Cone or Triangle being prolonged above its Vertex, this Section or the Circuit thereof is called an *Hyperbola*, ſuch is the Figure Z, formed in the Cone C, by a Plane dividing it according to the Line O. *See alſo P. B. P.* 8. *Fig*. 33.

THERE *are certain Terms which we neceſſarily uſe in treating of* Conic Sections, *which I thus explain*. Let the Section of a Cone thro' its Axis be the Triangle BAC. Let $AP=x$, $PM=y$, $AD=a$, $BD=b$. Then $x . y :: a . b$. Then $ay=bx$, which is the Equation, that expreſſes the Nature of the Triangle. Let AMB be a Semi-circle whoſe Center is N, and AN+NB, or AB is the Diameter. Let $AP=x$, $PM=y$, $AB=a$; hence $PB=a-x$. Wherefore $\div\div\ x .\ y .\ a-x$; therefore $ax-xx=yy$, is the Equation which expreſſes the Nature of the Circle. A Perpendicular ſuch as MP drawn from any Point whatſoever as M of the Circumference upon the Diameter AB, is called an *Ordinate*, and the Part of the Diameter taken between its Extremity and the meeting of the *Ordinate*, is called the *Abſciſſa*. Let DEE be a given right Line, with F a Point out of that Line, I call this Line DEE the *Directrix*. Let AMN be ſuch a Curve, that every Line Perpendicular upon DE falling upon M, one of the Points of the Curve, may have always one and the ſame Proportion with a ſecond Line drawn from the Point M to the given Point F. This Curve is regular and may be deſcribed *that is*, all the Points thro' which it ought to paſs may be found. To theſe Points, and theſe Lines, there are names given.

THE Point A is the *Vertex* of the Curve. The Point F is called the *Focus*. The Line AF produced is the *Axis*. The Lines which croſs the Axis Perpendicularly, and which are comprehended between the Curve and its Axis, are called *Ordinates*. PM is an *Ordinate*. The Part of the Axis taken from the Vertex A, to the Meeting of an Ordinate, is called *Abſciſſa*. The peculiar Ratio which is between the Line AD, to AF, is that which makes the eſſential Difference between theſe three Conic-Sections. The Parabola, Ellipſis and Hyperbola. In theſe three Sections I ſhall call the Proportion that of p. to q. In the Parabola, 'tis always a Ratio of Equality; in the Ellipſis p is always greater than q; and in the Hyperbola p is always leſs than q.

CHAP.

CHAP. II.

OF *the Parabola, or Curve-Line which repreſents the Section of a right Cone, by a Plane parallel to one of its Sides.*

DEFINITION I.

THE *right Line DE is given, with the Point F out of that Line. DE is the directrix of the Parabola. The Line FD is Perpendicular upon the Directrix DE. If this Perpendicular be divided at the Point A, ſo that FA=AD, and that having drawn from each Point of the Curve as M, a Perpendicular upon DE, and another Line at the Point F, if theſe two Lines are equal. MF=ME, this Curve is called a Parabola.*

IT will be demonſtrated that this Curve is the Conic-Section called a Parabola. I ſhall expreſs the Proportion of Equality of EM to MF, by the two Letters p and q.

PROB. 1. TO *find each Point of the Parabolic Curve.*

DE is the directrix Line which is given, and F the Focus. Divide DF into two equal Parts, hence A is the Vertex of the Curve. To find the other Points. 1. Thro' A draw a Parallel to DE, From which take AX=FA; and draw thro' D and X an indefinite Line. 2. Having afterwards drawn a ſufficient number of Lines parallel to DE, which cut the Axis DF, it being prolonged, the Points of theſe Parallels thro' which the Curve paſſes will be eaſily found; the Point M, *for Example*, in the Line PO. With the Extent of PO and from the Focus F as a Center, deſcribe a Circle which will cut PO in M; then AD. AX::p. q, and AD. AX::DP. PO. Now DP=ME, and by conſtruction PO=FM: Then * ME. FM::p. q. Hence M is one of the Points of the Parabola, according to the preceeding Definition. Obſerve that the indefinite Line DX makes with DF an Angle of 45 Degrees; for DAX is a right Angle, and AD=AX: Therefore the Triangle is Iſoceles; hence the Angle ADX is equal to AXD; conſequently each is 45 Degrees, or the Half of a right Angle.

DEF. 2. A *Line always double FD, or quadruple FA, or AD, is called the* Parameter of the Parabola.

LET the Parameter be called a; the Ordinate PM be called y, and the Abſciſſa PA be called x. There is no occaſion to be at the Trouble of explaining the Parameter any farther, than what is done by the Definition, which calls it thus, *a Line quadruple of FA or of AD.*

E e 2

---

* *B.* 3. *n.* 53.

THEOR. I. THE *Rectangle made of the Parameter and Abſciſſa, is equal to the Square of the Ordinate. Or that Ordinate is a mean Proportional between the Parameter and the Abſciſſa.*

LET FA or AD *(fig. preceed)* be called b. The Abſciſſa AP hath been called x. Then PD or $ME = b + x$, and $FP = x - b$, or $b - x$, according as the Point P is found above or below the Focus F. The Parameter a is Quadruple AD or AF, and therefore of b; hence $a = 4b$. It muſt be demonſtrated that $∺ a.\ y.\ x$; or that $ax = yy$. According to the firſt Definition $MF = EM$; but $EM = PA + AD = b + x$. Then $\overline{MF}^2 = bb + 2bx + xx$. And ſince that $FP = b - x$: Then $\overline{FP}^2 = bb - 2bx + xx$. Now $\overline{FM}^2 - \overline{EP}^2 = \overline{PM}^2 = yy$. * Then $bb + 2bx + xx - bb + 2bx - xx = yy$. But $+bb - bb = 0$, and $+xx - xx = 0$; then the Remainder is $2bx + 2bx$, or $4bx = yy$. Now $4b$ is equal to the Parameter a; therefore $ax = yy$ or $a.\ y :: y.\ x$; *which was to be proved.*

COROLLARY. THE *Squares of the Ordinates of a Parabola are one to another, as the Parts of the Axis taken between its Vertex and the Meeting of theſe ſame Ordinates.*

THE Parameter a is always the ſame; but x which is the Abſciſſa is leſſer or greater, according as the Ordinate is nearer or farther from the Vertex A. Now ſeeing that it is always $yy = ax$, in whatever place the Ordinate is taken; then yy or the Squares of the Ordinates are one to another as x, or the Abſciſſa's. †

THEOR. 2. THE *Curve or conic Section which has been defined and deſcribed, is a Parabola.*

SVT is a Cone cut by a plane parallel to its ſide TV, according to the line AF: it muſt be proved that the Section QAR, both is, and hath the Properties of the Parabola.

THE Lines PM and QF are perpendicular upon AF, which I call the Axis of this Curve. PM and QF are therefore Ordinates: Conſequently to prove that this Parabola QAR is that Curve which has been treated of, requires no more than to prove that the ſquares $\overline{MP}^2$ and $\overline{QF}^2$ and of all the other Ordinates, are one to another as the Parts AP, AF, of the Axis which they interſect; for this being ſo, then according to the preceeding Corollary, the Curve QAR is the line required. Let us ſuppoſe the right Cone SVT to be cut at the Point M by a plane Parallel to its Baſe. This Section PME is then a Circle. Let SQT be another Circle Parallel to DME. The Line MP is the Ordinate of the Parabola QAR, and of the Circle DME, it being perpendicular as well upon ED as upon AF, foraſmuch as it is the common Section of the Plane of the Circle and of that of the Parabola, which by Conſtruction, divides that of the Triangle at right Angles. It is the ſame of QF, which is alſo an Ordinate, to the Circle as well as to the Parabola. Now ∺ DP. MP. PE; and ſo likewiſe ∺ SF. FQ. TF: Then DP×PE=$\overline{MP}^2$, and SF×FT=$\overline{FQ}^2$: Then DP×PE. SF×FT :: $\overline{MP}^2$. $\overline{FQ}^2$; but PE=FT: Therefore $\overline{PM}^2$. $\overline{FQ}^2$::DP. SF. Now becauſe of the Similar Triangles DPA and SFA, AP. AF::DP. SF: then $\overline{PM}^2$. $\overline{FQ}^2$:: AP. AF: Therefore by the preceeding Corollary, QAR is of the ſame Nature as the Line treated of in that Chapter, which is alſo a *Parabola*.

---

* *B.* 4. *n.* 78. † *B.* 3. *n.* 54.

THEOR. .3 To *draw a Tangent thro' any Point required, of a given Parabola.*

LET AMB be a Semi-Parabola; required to draw the Tangent MT, from the Point M. Let there be drawn the Line FM from the Focus F to the Point M, and from the Point M the Line MP parallel to the Axis, and bounded by the Directrix DP. If there be drawn FP, and it be divided into two equal Parts, the Line MT drawn thro' the Point of Diviſion will be the Tangent required; which is to be proved. The Triangle EMP is Iſoceles; therefore MT is perpendicular upon FP. When a Line touches only in one Point it is a Tangent, which is the Caſe here; for every other Point that can be aſſigned in the Line TS, is not in the Parabola, but out of it; as for Example, N taken in TS above or below M, is out of the Parabola. For let there be drawn the Lines NF, NP, and NE, Parallel to the Axis, NP will be always greater than NE. Now ſince that TN is by Conſtruction perpendicular upon FP, the Lines NF and NP are equal. NF is then greater than NE. conſequently N is not one of the Points of the Parabola: for if it were, FN would be equal to NE, according to the Definition of the Parabola. But FN being greater, the Point N, is not in the Curve.

COROLLARY. IT *is evident that if the Curve AMB repreſent the Section of a parabolical Mirrour, and that CM be an Incident Ray parallel to the Axis AF, the Line MF will repreſent the reflected Ray; the Angle of Incidence CMS being equal to the Angle of Reflection FMT: For the Angle CMS is equal to the oppoſite Angle TMP. Now by Conſtruction, TMP is equal to TMF; therefore TMF is equal to CMS. Wherefore all the Rays which fall upon the Concave Surface of the Mirrour parallel to the Axis, will reunite at the Point F. and this occaſions it's being called the* Focus, *or burning Point. Hence the Concave parabolical Conoid, is the propereſt Form for a good burning Mirrour.*

CHAP.

CHAP. III.

OF *the Ellipſis, or Line which repreſents the Section of a Cone, by a Plane cutting its two Sides, and which is not parallel to that of its Baſe.*

DEF. 1. LET *there be given the right Line DE for the Directrix of the Curve that is to be defined, with the Point F; out of DE. FD is perpendicular upon DE. If this Perpendicular is divided at the Point A, according to the Ratio of p to q, I ſuppoſe p greater than q; and that having drawn a Perpendicular from each Point of the Curve, as M, upon DE, and another Line at the Point F, if the Line EM be always to FM as p to q; this Line ought to be called an Elipſis, becauſe it will be proved to be the ſame as the Section of the Cone which is called by that Name.*

PROB. 1. THE *Directrix DE being given with F, which is a Point out of it, to find all the Points of the Ellipſis.*

1. FD muſt be divided in A; ſo that AD. FA :: p. q. 2. Thro' A draw a Parallel to the Directrix DE. and take AX equal to AF. Hence AD. AX :: p. q. 3. Thro' D and X draw an indefinite Line; after which dividing the Axis by as many Parallels as you pleaſe, it will be eaſy to find in thoſe Parallels, the Point thro' which the Curve paſſes. From F as a Center, and with the Extent of PO, deſcribe a Circle which will cut PO in M. By reaſon of the ſimilar Triangles DAX and DPO, the Line PD, or its equal EM is to PO or FM, as AD is to AX; therefore EM. FM::AD. AX::p. q. The Point M is therefore a Point of the elliptical Curve.

PROB. 2. To *find the Axis of this Curve.*

LET the Line DX be prolonged *ad inf.* and upon the produced Part of AD at the Point F let there be drawn a Line making an Angle of 45 Degrees, and continued even till it meets the indefinite Line DX. From the Point of Interſection x, let there be drawn the Perpendicular xa upon Fa. Then in the Triangle Fax, the Angle Fxa is alſo of 45 Degrees: therefore the Triangle Fax, is Iſoceles. Now AD. AX::Da. ax; and ſeeing that Fax is Iſoceles, ax=Fa: Then AD. AX::Da. Fa. But AD. AX::p. q. Then Da. Fa::p. q. Wherefore a is one of the Points of the Curve, as is evident.

DEF. 2. THE *tranſverſe or longeſt Axis of an Ellipſis is Aa. The Point C which divides Aa into two equal Parts, is the Center. The Line BG which cuts Aa at right Angles, is the conjugate or leſſer Axis. Having taken fa equal to FA, the Points F and f are the Focii. 2. The Perpendiculars drawn from any Point whatſoever of the Curve to one of the Axes, are called* Ordinates.

HENCE PM being an Ordinate to the Axis *Aa*, the Line AP is an Abſciſſa. 3. The third Proportional to the two Axes is called the Parameter of the firſt Term of the Proportion.

THEOR. 1. THE *Diſtance Ff of the Focii F and f, is to the greater Axis* Aa, as *q* is to *p*. (*fig. preceed.*)

IT has been demonſtrated that $q.\ p::AX.\ AD::Fa.\ Da$: Then $Fa-AX.\ Da-AD::q.\ p.$ Now ſince that $fa=FA=AX$: Then $Fa-AX=Ff$. But $Da-AD=Aa$: Therefore $Ff.\ Aa::q.\ p$; *Q. E. D.*

THEOR. 2. *AC the Half of the greater Axis Aa, is a mean Proportional between FC, the Half of the Diſtance of the Focii, and the Line CD.*

THE Half is to the Half, (*fig. preceed.*) as the whole is to the Whole; wherefore $AC.\ FC::Aa.\ Ff$. But by the preceeding Theorem, $Aa.\ Ff::AD.\ AX$. Then $AC.\ FC::AD.\ AX$; or $FC.\ AC::AX.\ AD$, and $FC.\ FC+AX::AC.\ AC+AD$. Now by conſtruction $FA=AX$; hence $FC+AX=FC+FA=AC$, and $AC+AD=CD$ Putting then AC in the Room of $FC+AX$ and CD in the Place of $AC+CD$, it will be $FC.\ AC::AC.\ CD$. Wherefore $\div$ FC. AC. CD; *Q. E. D.*

PREPARATION.

FOR *the Demonſtrations of the following Theorems.* (ſame Figure.)

LET $AC=d$, and $CF=g$: Then FA or $AX=d-g$, and Fa or $ax=d+g$. Now AC or d is a mean Proportional between CF or g and CD, by the preceeding Theorem. Then $\div\ g.\ d.$ CD. Then d multiplied by d, and the Product dd divided by g, the Quotient $\frac{dd}{g}$ is equal to CD. Let $CP=x$, and $PM=y$: Then $PF=x-g$, or $g-x$, and DP, or $ME=\frac{dd}{g}-x$, when the Point P is above the Center C. By the Generation of the Ellipſis. $d.\ g::\frac{dd}{g}-x.\ MF=d-\frac{gx}{d}$, * the fourth Term of this Proportion is MF. When the Point P, is below the Center C, then $PF=x+g$, and DP, or $ME=\frac{dd}{g}+x$: Then, as above, $MF=d+\frac{gx}{d}$.

THEOR. 3. THE *Square of any Ordinate whatſoever as PM at the greater Axis Aa, is to the Rectangle AP×Pa, the Parts of that Axis, as the Rectangle AF×Fa, Parts alſo of that Axis,*

---

* *B.* 3. *n.* 60.

is to the Square of AC or Ca, the Half of that ſame Axis. (ſame Fig.)

$AC = d$, $CF = g$, $PC = x$: Then $AP = d - x$, and $Pa = d + x$. Hence $AP \times Pa = dd - xx$. So likewiſe ſeeing that $AF = d - g$, and $Fa = d + g$: Then $AF \times Fa = dd - gg$. Hence ſuppoſing $PM = y$, it is then required to demonſtrate that $yy.\ dd - xx :: dd - gg.\ dd$. Now as the Product of the Extremes is equal to that of the Means, *that is*, $yydd = d^4 - ddxx - dd\,gg + gg\,xx$. The Triangle FMP being right angled $\overline{FM}^2 - \overline{PF}^2 = \overline{PM}^2 = yy$. Now $FM = d - \frac{gx}{d}$; hence $\overline{FM}^2 = dd - 2gx + \frac{ggxx}{dd}$, and $PF = g - x$: Then $\overline{PF}^2 = gg - 2gx + xx$. Then $\overline{PM}^2$, or $yy = dd - 2gx + \frac{ggxx}{dd} - gg + 2gx - xx$. Now cancelling the Terms which deſtroy one another by + and —, it will be $yy = dd + \frac{ggxx}{dd} - gg - xx$; and all the Terms on both Sides multiplied by dd, it will be $ddyy = d^4 + ggxx - ggdd - ddxx$; then &c.

THEOR. 4. LET *BC*$= b$, *the Rectangle AF×Fa, or* $dd - gg$ *is equal to bb the Square of BC, half the leſſer Axis BG.* (ſame Figure.)

IT muſt be proved that $bb = dd - gg$, $BC = b$ is an Ordinate which is at the Center; hence $x = o$, and $d - x = d$, and $y = b$; hence $yy = bb$. Now by the preceeding Theorem $yy.\ dd - xx :: dd - gg.\ dd$. Then ſubſtituting bb inſtead of yy, and cancelling $-xx$, it will be $bb.\ dd :: dd - gg.\ dd$; or $bb.\ dd - gg :: dd.\ dd$; hence $bb = dd - gg$; *which was to be proved.*

THEOR. 5. THE *Square of MK an Ordinate of the leſſer Axis BG is to the Rectangle BK×KG parts of that Axis, as the Square of the greateſt Axis Aa is to the Square of the Leſſer BG.* (ſame Figure.)

THE greater Axis A*a* is equal to $AC + Ca$, or to $2AC$; therefore to $2d$. The Half of the Leſſer Axis BG is called b; hence $2b = BG$. The Line CP has been already called x. Now the Ordinate MK is equal thereto, which is therefore x. The Line PM the Ordinate of the greater Axis hath been called y. Hence CK which is equal thereto, is alſo y; wherefore $b - y = BK$, and $b + y = KG$, and $bb - yy = BK \times KG$. The Square of the greater Axis A*a* or $2d$, is $4dd$. That of the leſſer Axis BG or $2b$, is $4bb$. Therefore this is what is to be demonſtrated. $xx.\ bb - yy :: 4dd.\ 4bb$. According to the third Theorem, $yy.\ dd - xx :: dd - gg.\ dd$. By the Fourth $bb = dd - gg$. Then Subſtituting bb inſtead of $dd - gg$, it becomes $yy.\ dd - xx :: bb.\ dd$. Then $yy.\ dd - xx :: 4bb.\ 4dd$; and the Extremes and Means multiplied, it will be $4ddyy = 4bbdd - 4bbxx$, or $4bbxx = 4bbdd - 4ddyy$; and turning this Equation into a Proportion, it will at laſt be $xx.\ bb-yy :: 4dd.\ 4bb$.

THEOR. 6. THE *Parameter is to the Diameter as the Square of its Ordinate; is to the Rectangle comprehended under the Parts of the Diameter taken from the Section of that Ordinate.* (ſame Figure.)

AC the Half of the longeſt Diameter Aa has been called d; and b that of the ſhorteſt Diameter BC. Hence BG$=2b$. Let the Parameter be called a; then according to the Definition, $2d.\ 2b :: 2b.\ a$; then $\frac{4bb}{2d}$ or $\frac{2bb}{d}=a$, and both Members of this Equation multiplied by d, it will be $2bb=ad$, and divided by 2, it is $bb=$half ad. By the third and fourth Theorems above, it appeared that $yy.\ dd-xx :: bb.\ dd$. Putting half ad in the Room of bb, it will be $yy.\ dd-xx ::$ half $ad.\ dd$. Each Term of the Ratio of half ad to dd divided by d; gives half a, and d; and multiplying them by 2, it becomes a, $2d$; therefore $yy.\ dd-xx :: a.\ 2d$, or $a.\ 2d :: yy.\ dd-xx$; *which was to be proved.*

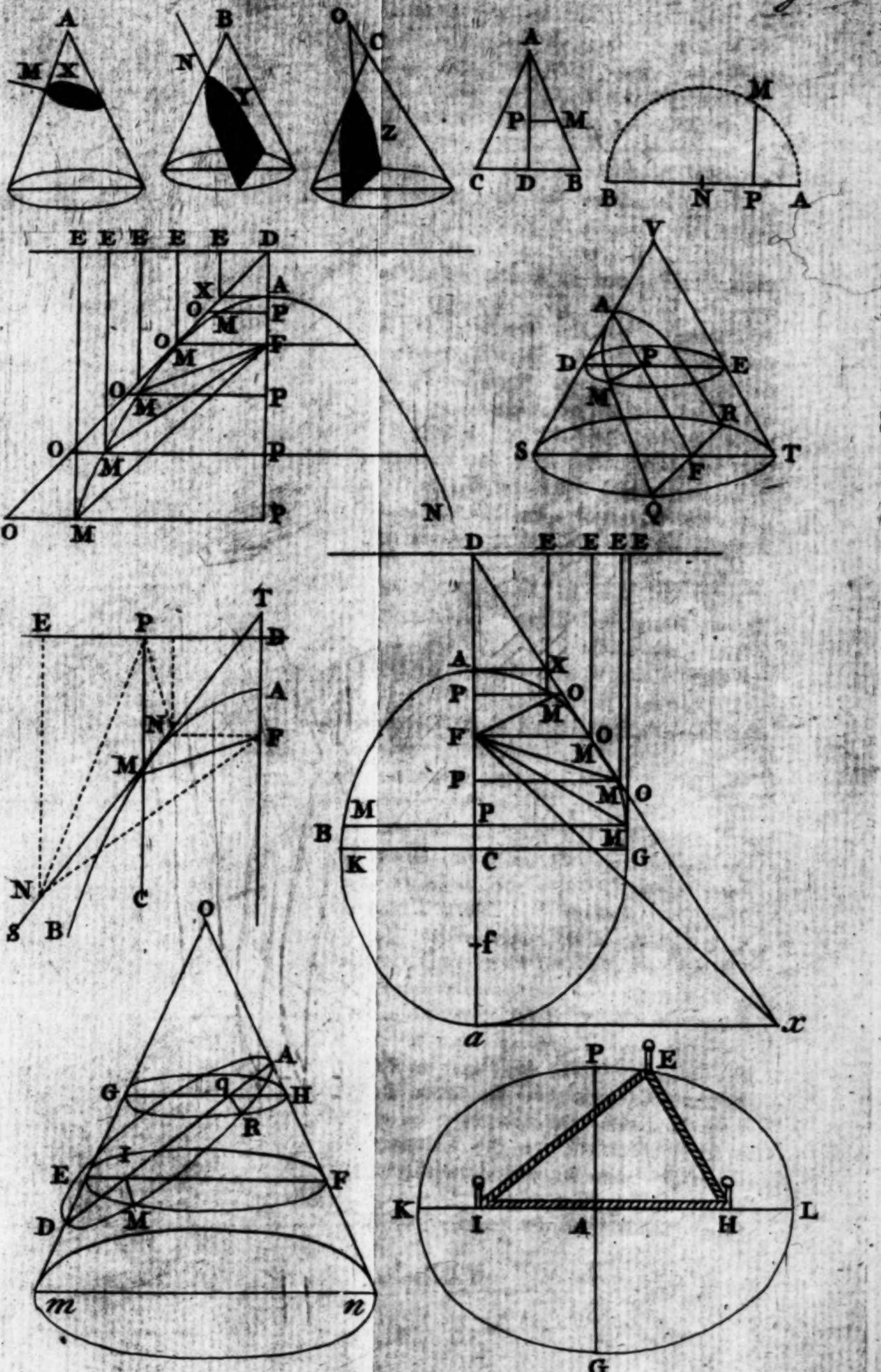
THEOR. 7. THE *Squares of the Ordinates are one to another, as the Rectangles of the Parts of the Axis made by the Meeting of the ſame Ordinates.*

Y and m are two different Ordinates, and n a Part different from PC. By the preceeding Theorem, $a.\ 2d :: yy.\ dd-xx :: mm.\ dd-nn$. Therefore $yy.\ mm :: dd-xx.\ dd-nn$; *which was to be proved.*

THEOR. 8. THE *Conic Section or Curve which has been defined and deſcribed, is an Ellipſis.*

LET the Cone mon be cut by a Plane making an oblique Angle with its Axis, and which interſects both its Sides. The Section DMAD, which is an Ellipſis, is the ſame Line as that whoſe Properties we treated of. Let there be conceived two parallel Circles EMF and GRH, whoſe Diameters interſect that of the Curve DMAD, in I and in q. If from the Points M and R, where thoſe Circles croſs that Curve, there be drawn right Lines to the Points I and q; it is evident that theſe Lines are Ordinates common to both the Circles and to the Curve DMAD, foraſmuch as Rq is Perpendicular as well upon HG as upon AD, being the common Section of the Plane of the Circle and of the Ellipſis, which conſequently interſects that of the Triangle at right Angles. It is the ſame of MI. The Triangles AqH and AIF are Similar; then Aq. FI :: Aq. AI. The Triangles DEI and DGq, are alſo Similar; then Gq. EI :: qD. ID. Each Term of the firſt Proportion multiplied by its correſponding Term in the Second, the Products being compounded of equal Ratio's, are Proportional. Hq. FI :: Aq. AI, Gq. EI :: Dq. DI. Hence H$q\times$Gq. FI$\times$EI :: A$q\times$Dq. IA$\times$DI, becauſe of the Circles EMF and GRH, according to theſe Elements $\overline{qR}^2=Hq\times qG$, and $\overline{IM}^2=FI\times IE$: Then $\overline{qR}^2$, $\overline{IM}^2$; A$q\times$

Conic Sections. Plate. 1.



$fM+ME$; conſequently longer than $fM+MF$: For by conſtruction $MF=ME$; therefore the Point D cannot be in the Ellipſis, which hath by the preceeding Theorem this property, that if D be one of its Points, $fD+DF=fM+MF$. The Line DT cannot therefore touch the Ellipſis, but at the Point M.

COROLLARY. It *is evident that if AMB repreſent the Section of an Elliptical Mirrour, and fM be an incident Ray, it will be reflected at F.*

FOR the Angle of Incidence fMS is equal to the Angle of Reflexion FMT; ſince that the Angle SMf is equal to EMT, which is equal to TMF. Hence all the Rays which diverge from *f*, and which fall upon the concave Surface of the Mirrour, will reunite at the Point F, and reciprocally if they diverge from the Point F, they will reunite at the Point *f*: For this reaſon it is that theſe Points are called the *Focii*.

C H A P. IV.

OF *the Hyperbola, or Line which repreſents the Section of a Cone cut by a Plane parallel to its Axis, or ſo that cutting only one Side of the ſaid Cone, it might alſo cut the other if prolonged above its Vertex.*

DEF. 1. LET *DE be a Line taken for the Directrix, and F a Point out of that Line. The Line FD is Perpendicular upon DE. If FD is divided at the Point A, according to the Ratio of p to q,* (it is here ſuppoſed that *p* is leſſer than *q*) *and that a Curve is traced, from which having drawn Perpendiculars to DE as EM, and other Lines to F as FM, ſuch that EM may be to MF as p to q, this Line is called an* Hyperbola; *becauſe as is demonſtrable it hath the ſame Properties as the Conic Section of that name.*

PROB. 1. To *find all the Points thro' which this Curve paſſes.*

DRAW AX Parallel to DE, and equal to FA, and draw an indefinite Line on each Side thereof thro' the Points D and X. After which drawing freely as many Parallels to DE as you pleaſe, as PO: to find the Point of this Line PO thro' which the Curve paſſes, deſcribe a Circle from F as a Center, with the Extent FO. The Point M, where the Circle cuts PO, is the Point required; for the Triangles DAX and DPO being ſimilar, $PO.\ PD :: AX.\ AD :: q.\ p$. Now $PO=FM$ by conſtruction. And $PD=EM$: Then $FM.\ EM :: q.\ p$. The Point M is therefore in the Hyperbola, by the preceeding Definition.

PROB. 2. To *find the Vertex of a Similar, oppoſite Curve.*

IN conſidering the Properties of this Curve, it may be compared with another oppoſite ſimilar Curve whoſe Vertex is found thus. Upon FD make the Angle DFx of 45 Degrees, and from the Point x where Fx cuts the indefinite Line XD, draw a Perpendicular upon Ff. The Point a upon which this Perpendicular falls, is one of the Points of the Curve: For Fax being Iſoceles, $ax=aF$. The Triangles xDa and ADX are Similar, by conſtruction; then $aD.\ ax$ or $aF :: DA.\ AX$ or AF. Hence $aD.\ aF :: DA.\ AF :: p.\ q$: Then $aD.\ aF :: p.\ q$. Hence the Point a is one of the Points of an oppoſite and ſimilar Curve. Having taken $fa=FA$, it gives the Focus f, according to the following Definition.

DEF. 2. THE *Line* Aa *is called the* Tranverſe *or* prolonged Axis. *The Point* C *which divides the tranſverſe Axis in two, is called the* Center. *If thro' C be drawn a Perpendicular BB, whoſe length is bounded by a Circle deſcribed from the Point A, and with the Extent CF, that Line BB is called the* conjugate Axis. *The Points F and f are the* Focii. *The Perpendiculars drawn from one of the Points of the Curve upon one of the Axes, are* Ordinates. *The* Abſciſſa's *are the Parts of the Axes taken from their beginning to the meeting of the Ordinates. Sometimes their Origin is at the Center, and ſometimes at the Vertex of the* Hyperbola's. *The third Proportional to the two Axis is called the* Parameter *of the firſt Term of the Proportion.*

THEOR. 1. THE *tranſverſe Axis* Aa *is to* Ff, *the Diſtance of the Focii, as* p *to* q.

Dax and DAX are two ſimilar Triangles: *(fig. preceed.)* then $AX.\ ax :: AD.\ Da$. Now $AX=AF$, and $ax=aF$: Then $AF.\ aF :: AD.\ Da$, then *compoundedly* $AD+Da.\ AD :: AF+Fa.\ AF$. But $AF+Fa=Ff$: For $AF=af$, and $AD+Da=Aa$. Then $Aa.\ AD :: Ff.\ AF$, and $Aa.\ Ff :: AD.\ AF$. Now $AD.\ AF$, or $AX :: p.\ q$; therefore $Aa.\ Ff :: p.\ q$; *which was to be proved.*

THEOR. 2. CD *is a third Proportional to CA and CE.* (preceed fig.)

SEEING that the Wholes are to the Wholes, as the Halves are to the Halves, as CF is the Half of Ff, and CA the half of Aa: Then $Ff.\ Aa :: CF.\ CA :: AX.\ AD$: Then $CF.\ CA :: CF-AX$, or $AF.\ Ca-AD$. Now $CF-AF=CA$, and $CA-AD=CD$. Therefore $CF.\ CA :: CA.\ CD$; *which was to be proved.*

THEOR. 3. THE *Square of any Ordinate whatſoever as PM of the tranſverſe Axis Aa, is to the Rectangle of* $AP \times pa$ *the Parts of that Axis, as the Rectangle* $AF \times Fa$ *is to the Square of CA or Ca.*

LET $CA=d$ the Ordinate $PM=y$, $CP=x$, and $CF=g$; *(fig. preceed.)* when P is above or below the Point F, $PF=x-g$, or $g-x$. Then it will be $DP=x-\frac{dd}{g}$; for by the preceeding Theorem $CD=\frac{dd}{g}$. But by conſtruction and by the firſt Theorem,

d.

$d.\ g :: x - \frac{dd}{g}.\ MF$; then $MF = \frac{gx}{d} - d$. If the Point P were above the Point *a*, then $MF = \frac{gx}{d} + d$. It muſt then be proved that *yy* or $\overline{PM}^2.\ xx - dd$, or $AP \times Pa :: gg - dd$, or $AF \times Fa.\ dd$, or $\overline{CA}^2$, or $\overline{Ca}^2$; which is eaſy: For upon account of the right angled Triangle MPF, $\overline{PM}^2 = \overline{FM}^2 - \overline{PF}^2$, *that is*, in Analytical Terms, $yy = \frac{ggxx}{dd} - xx - gg + dd$; or by multiplying both Sides by *dd*; it will be $yydd = ggxx - ddxx - ggdd + d^4$, and reducing the Terms compoſing this Equation into a Proportion, it is $yy.\ xx - dd :: gg - dd.\ dd$; *Q. E. D.*

THEOR. 4. THE *Square of any Ordinate whatſoever as PM of the Axis Aa is to the Rectangle of the Parts of that Axis* (viz. to $AP \times Pa$,) *as the Square of the conjugate Axis BB is to the Square of the other Axis Aa.* (ſame Figure.)

LET us conceive a right angled Triangle ABC; according to the Definition of the conjugate Axis, $AB = FC$, it has been ſuppoſed $FC = g$, and $AC = d$; let us ſuppoſe $BC = b$. Then ABC being right angled, and $AB = g$; then $bb = gg - dd$. Now by the preceeding Theorem, $yy.\ xx - dd :: gg - dd$, or $bb.\ dd$. Multiplying *bb* and *dd* by 4, they remain in the ſame Ratio. Then $yy.\ xx - dd :: 4bb.\ 4dd$. Now $AP \times Pa = xx - dd$. Therefore the Square of *yy* is to the Rectangle $AP \times Pa$, as $4bb$, the Square of the conjugate Axis BB or $2b$, is to the Square $4dd$ of the other Axis A*a* equal to $2d$.

THEOR. 5. THE *Parameter is to the Diameter as the Square of any Ordinate whatſoever, is to the Rectangle of the Parts of that Axis made by the meeting of that Ordinate.*

LET the ſame Denomination of the Parts be preſerved and the ſame Figure as before, and let *m* be the parameter of the Axis A*a*, which is by its definition a third Proportional to the ſaid Axes; hence $\div m.\ 2b.\ 2d$. But by reaſon of this Proportion it will be $m.\ 2d :: 4bb.\ 4dd$. Now by the preceeding Theorem, $4bb$ is to $4dd$ as *yy* the Square of the Ordinate is to $xx - dd$ the Rectangle of the Parts of the Axis; therefore $m.\ 2d :: yy.\ xx - dd$, *Q. E. D.*

THEOR. 6. THE *Square of the Ordinates are one to another as the Rectangles of the Parts of the Axis, made by the meeting of the ſame Ordinates.*

WHICH is evident, ſince that in what place ſoever the Point P is found, the Parameter *m* is to the Diameter $2d$, as the Square *yy* of the given Ordinate is to the Rectangle $xx - dd$, (*or* $AP \times Pa$) made of the Parts of the Axis limitted by the Ordinate.

Theorem

Theor. 7. The Hyperbola *is the ſame Curve or Conic Section as that which has been deſcribed.*

Let *m* O *n* be a Cone cut by a Plane, which alſo meets the other Side *n* O, of the Cone in B, if prolonged above the Vertex O. This Section *q* AR is the Curve that has been treated of, or this Curve is really that Section of the Cone which is called *Hyperbola*. Let a Circle DME be conceived Parallel to *m q n*, that of the Baſe. Their Diameters cut the Axis of the Hyperbola at the Points P and F. The Lines MP and F *q* are Perpendicular upon thoſe Diameters as well as upon the Axis AF, foraſmuch as they are the common Sections of two Planes perpendicular to that of the Triangle; conſequently they are as much Ordinates of the Circle, as of the Curve DME. Now $\overline{MP}^2 = DP \times PE$, and $\overline{Fq}^2 = mF \times nF$. It is to be demonſtrated that $\overline{MP}^2$, or DP×PE is to $\overline{Fq}^2$, or its equal *m*F×*n*F, as AP×PB is to AF×FB. The two Triangles DAP and *m* AF are Similar. The two Triangles FB *n*, and PBE are Similar. Therefore *m* F. DP ::FA. PA. Alſo *n*F. PE::FB. PB. Each Term of the firſt Proportion multiplied by its correſponding Term in the Second, the Products are in Proportion, they being Rectangles whoſe Ratio's are compounded of ſimilar Sides. DP×PE. *m*F×*n*F::AP×PB. AF×FB. Then ſeeing that $\overline{MP}^2 = DP \times PE$, and $\overline{Fq}^2 = mF \times nF$. $\overline{MP}^2$. $\overline{Fq}^2$::AP×PB. AF×FB; *which was to be proved.*

SCHOLIUM,

If *the Side of the Cone be prolonged, ſo that there be ſuppoſed another oppoſite Cone thro' the Vertex thereof, it is plain, that if the interſecting Plane be produced, it will form in the other Cone a ſimilar Section, which is an oppoſite Hyperbola.*

Prob. 3. To *deſcribe an Hyperbola by a continued motion.*

Let AMX be an Hyperbola, its Vertex is A, and its Focus F. The Vertex of its oppoſite Hyperbola is B, whoſe Focus is *f*. This Hyperbola may be conceived as already deſcribed, or it may be deſcribed thus. The Line C*f* cuts the Hyperbola AMX at the Point M. Let us conſider this Line C*f*, as a Ruler at the End of which is faſtened a Cord whoſe other Extremity is fixed to the Focus F. This Ruler is faſtened by its other End to the Focus of the oppoſite Hyperbola, and may be turned about its Point *f*. Let us conceive this Ruler at firſt, as laid upon AB, and the Cord of ſuch a length that M may then coincide with A. Put a Finger at the Point A or M, and permit it to go towards X, by turning about the Ruler; but always keeping the Chord extended and as fixed againſt the Ruler. Whilſt the Ruler turns about, the Finger conſidered as a Point, will deſcribe the Hyperbola AMX.

Corol. 1. From *this Conſtruction it follows, that having drawn from each Point of the Hyperbola, two right Lines to the two Focii F and f, the Difference between thoſe two Lines will be always the ſame.*

In its firſt Poſition where M is upon A, the Ruler upon AB or upon F*f*, it is evident that the Difference between M*f* and MF, *that is*, M*f*–MF, is AB, ſince that FA=*f*B; this difference is always the ſame in the other Poſitions. For while the Ruler turns about, and as the Finger moves, the Cord MF and the Part M*f* of the Ruler encreaſe their length equally. Therefore theſe two Quantities preſerve the ſame Difference.

Corol. 2. An *Hyperbola may be extended or continued, ad inf.*

For if the Ruler C*f* be prolonged *ad inf.* and at the ſame Time the Cord MC, making the Ruler turn, and continuing to preſs the Cord as above, it will prolong the Hyperbola without end. It is evident that the Point M will diverge more and more from the Focus F, whilſt the Ruler *f*C is turning about.

SCHOLIUM.

The *Hyperbola being prolonged and continued* ad inf. *as aforeſaid, it will incline more and more to a certain right Line, without ever meeting it, which right Line is called the* Aſſymptote *of the Hyperbola; which is a Greek Word, ſignifying that property.*

Prob. 4. To *draw the Tangent MT, from any Point whatſoever of an Hyperbola.*

Let there be two oppoſite Hyperbola's A and B. From the given Point M draw the Lines MF, M*f*, to the Focii F and *f*, and divide the Angle FM*f* into two equal parts by the Line MT; I ſay that it is a tangent Line. Let ME be taken equal to MF. Alſo from any Point whatſoever as D in the Tangent MT, let there be drawn the Lines DF, D*f*, DE. The Triangles MFN and MEN are equal; as alſo NDF and NDE. It is required to prove that the Point D which is in the Tangent MT, is not in the Hyperbola; and that therefore this Tangent does neither touch, nor meet in D. If D were in the Hyperbola, D*f*–DF =M*f*–MF by the preceeding Corollary. Now 'tis not ſo; for ME being taken equal to MF, then M*f*–MF=E*f*. And ſince that DE is equal to DF, then D*f*=E*f*+DF. But in the Triangle DE*f*, the Sides DE and E*f* taken together, are longer than the Side D*f*; therefore DE exceeds DF; then the Point D is not one of thoſe of the Hyperbola; which may be ſaid of every other Point of the Line MT, taken either above or below M; therefore this Line MT can meet the Hyperbola whoſe Tangent it is, only in one Point M.

Corollary. If *the Curve AM repreſent the Section of an Hyperbolical Mirrour, and that CM be an incident Ray which falls upon*

upon the concave Surface of the Mirrour at the Point M, it will therefore reſlect from C *to* F, *and go towards f, the Focus of the oppoſite Hyperbola.*

The Angle DMC is equal to the Angle EMN, by conſtruction equal to NMF; hence DMC is equal to NMF. The Angle of incidence CMD being then equal to the Angle of reflexion FMT, the Ray CM will be reflected in F to the Focus of the Hyperbola. Which ought to be underſtood of every other incident Ray: and for this reaſon it is that the Points F and *f*, are called Focii.

These *Curves have innumerable other Properties, which may be ſeen in an excellent Treatiſe of* Conic Sections, *of the late Marquis* de L'Hoſpital.

The Engliſh reader may have Recourſe to the Tranſlation of this Work, by Edmund Stone, *F. R. S.*

G.R. III
BRITISH MUSEUM

* * *

FINIS.

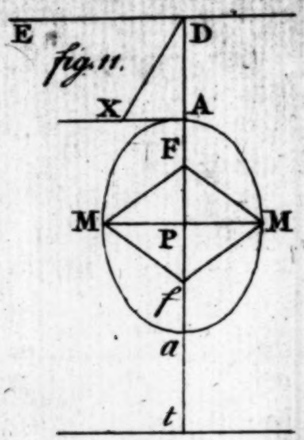


fig. 11.

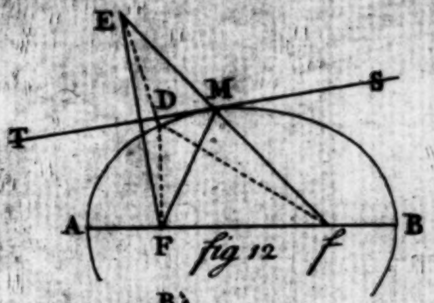


fig 12

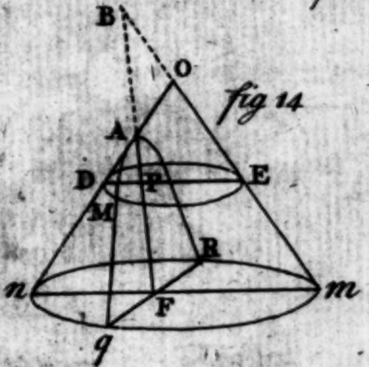


fig 14

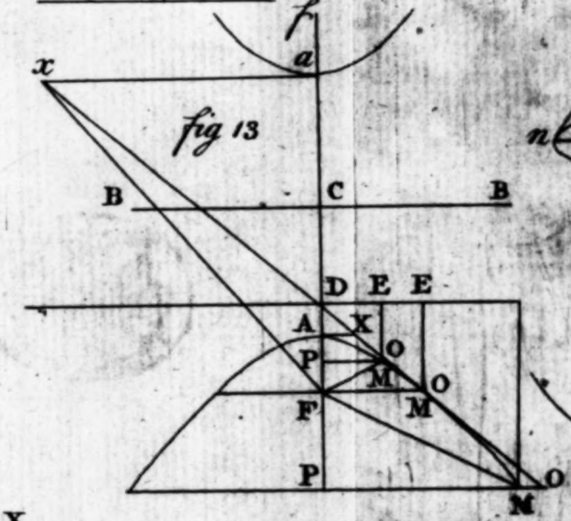


fig 13

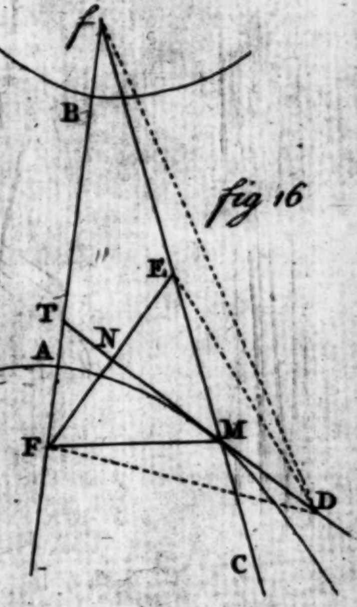


fig 16

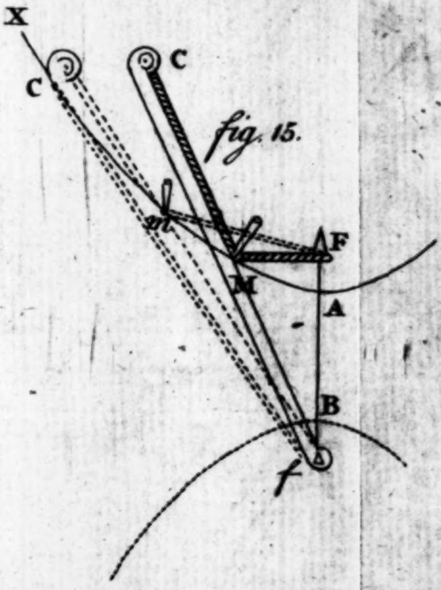


fig. 15.

ADVERTISEMENT.

THE Method of folding together the Schemes contained in the following Plates, is ſo obvious that I apprehend but very little need be ſaid concerning it; ſhall therefore only obſerve, that if each of their Sides or Faces be firſt raiſed up and preſſed down flat, with the Finger, according to the Direction of the Lines which are cut half thro' on the back ſide, it will occaſion the Solid to form more readily. The Part which is ſhaded is the Baſe, or part round which the reſt are to be folded; except *fig.* 15, *Plate* 5, where the ſhade is miſplaced, as appears by the cutting of that Figure. As to thoſe Figures in *Plate* 9 and 10. which have ſome of their Lines half cut thro' on one ſide, and ſome on the other, it will be proper to begin with folding thoſe Parts firſt which are neareſt the Baſe, or part which remains fixed, obſerving when you come to a black Line which is cut half thro' to fold it back, ſo that the two Sides may lie cloſe together, by which means the ſeveral Solids which the whole is divided into, will cloſely adhere to each other, each appearing diſtinctly, and which altogether will compoſe the whole Solid itſelf, according to its real form before it was ſo divided.

Shortly will be publiſhed an additional Plate conſiſting of two Sides, each of the ſame ſize with thoſe contained in this Work, containing the real Inſcription of the five regular Solids, treated of in Euclid's fifteenth Book, each body being properly cut, from Schemes which are ſo adapted that they plainly exhibit the actual Inſcription, and Circumſcription of each. Price One Shilling.

G.R. III
BRITISH MUSEUM

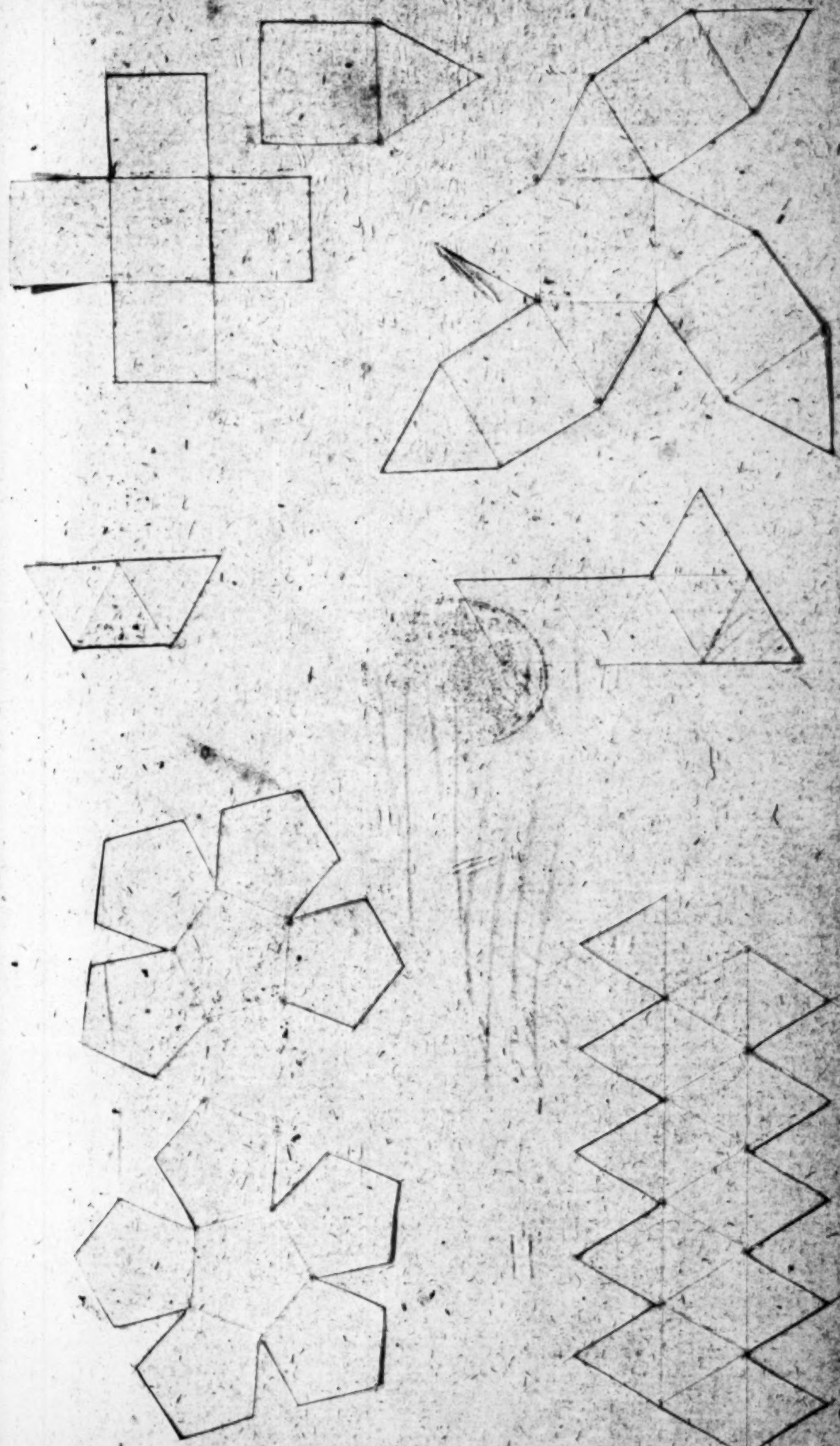
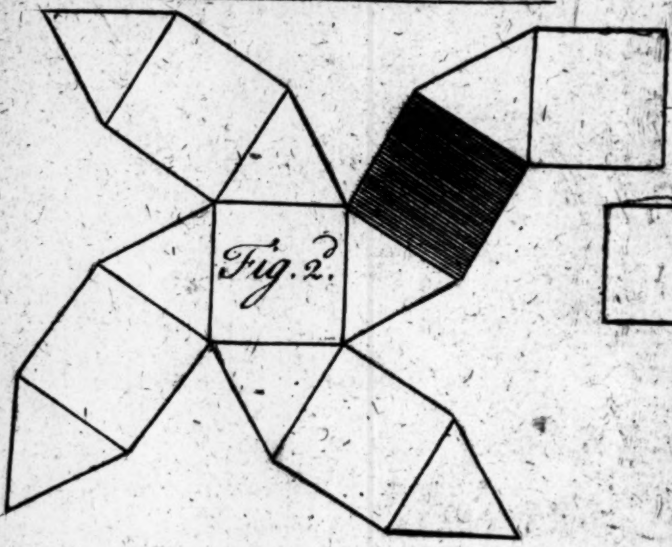
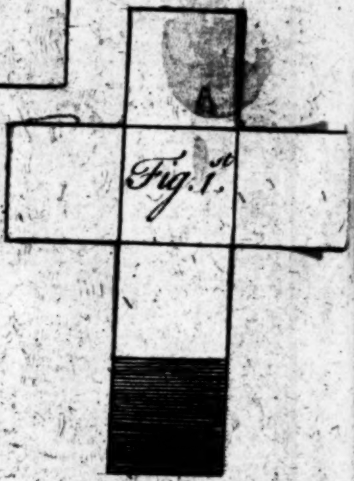


Plate. 1.

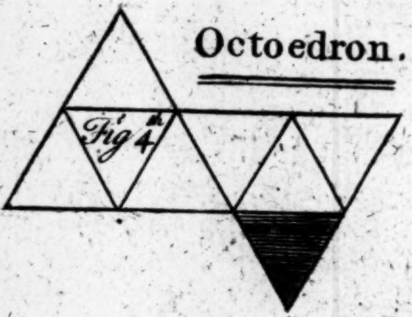
Exoctoedron *or* Canted Cube.



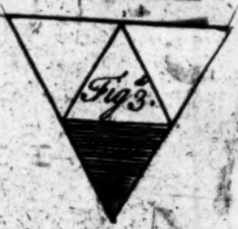
Hexaedron *or* Cu



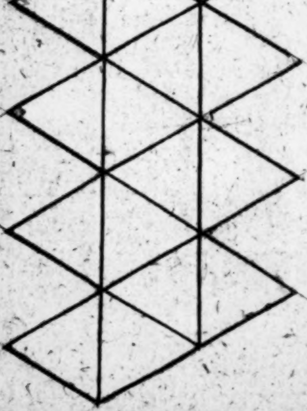
Octoedron.



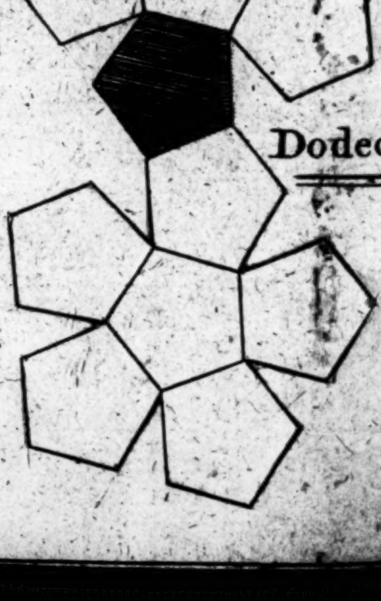
Tetraedron.



Icoſaedron.



Dodecaedron



Icoſidodecaedron.

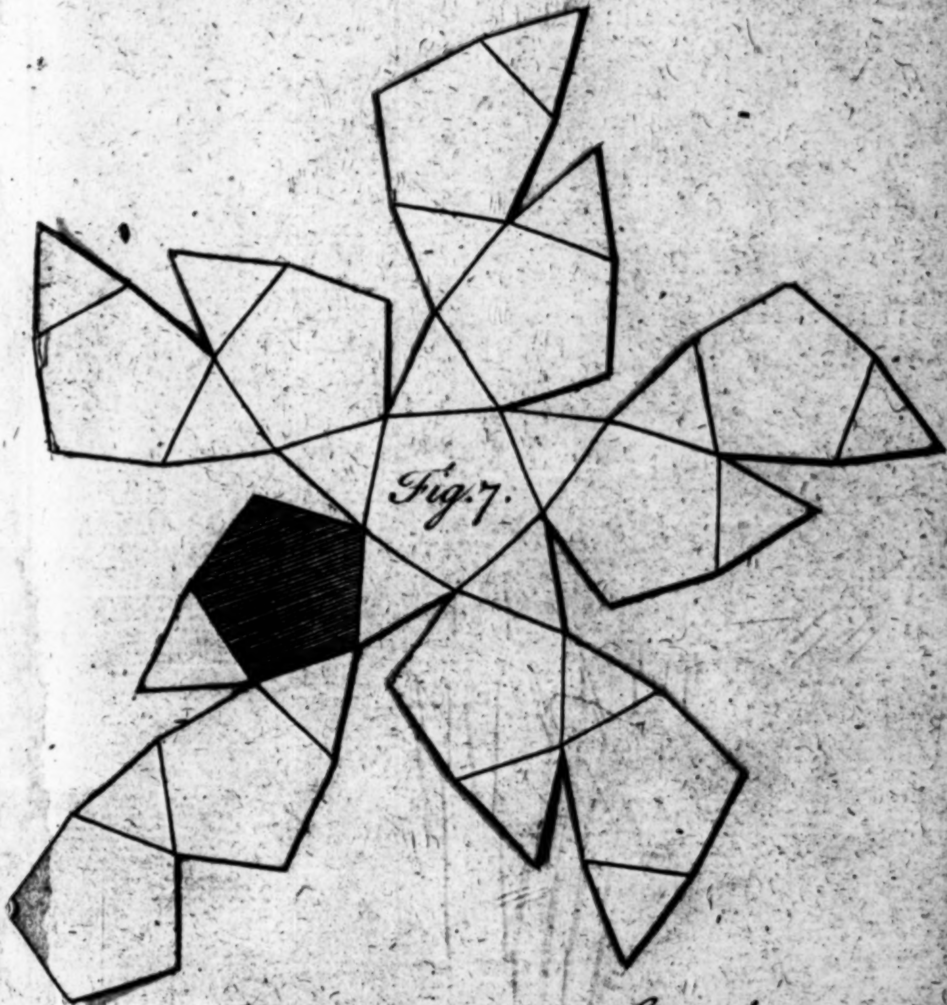


Fig. 7.

Hexaedron, *transform'd.*

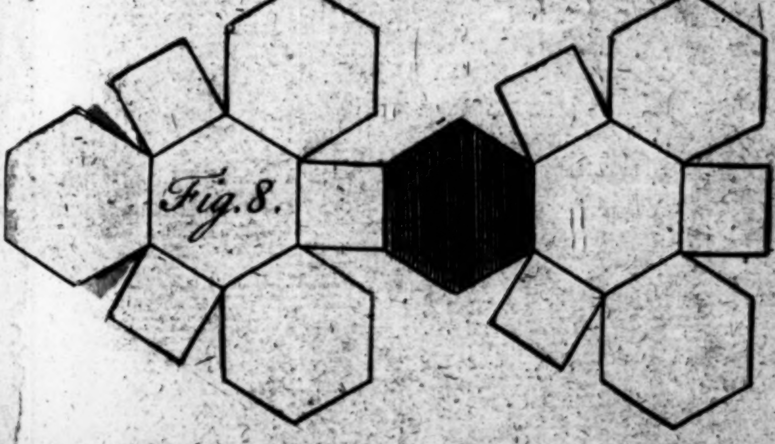
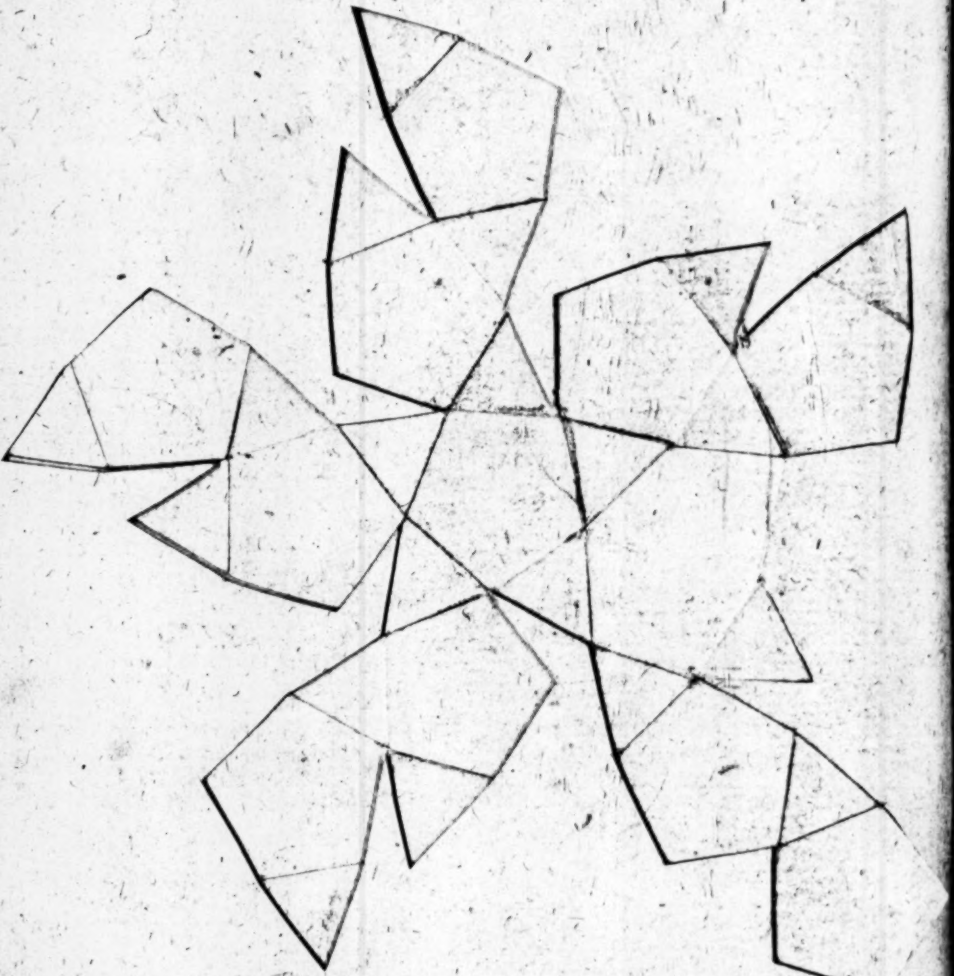
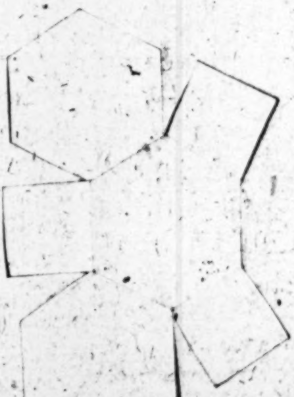
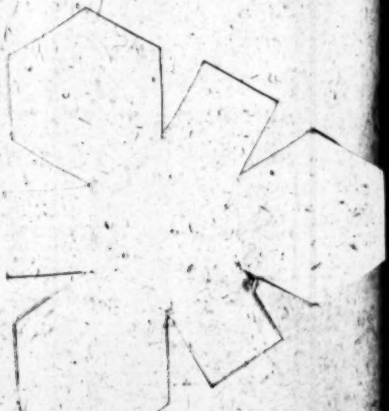


Fig. 8.







Tetraedron,
transform'd.

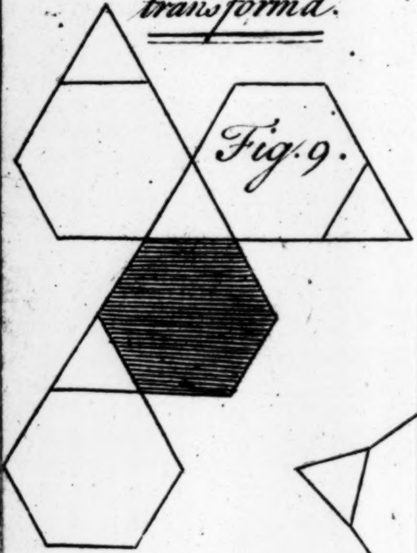


Fig. 9.

Octoedron,
transform'd.

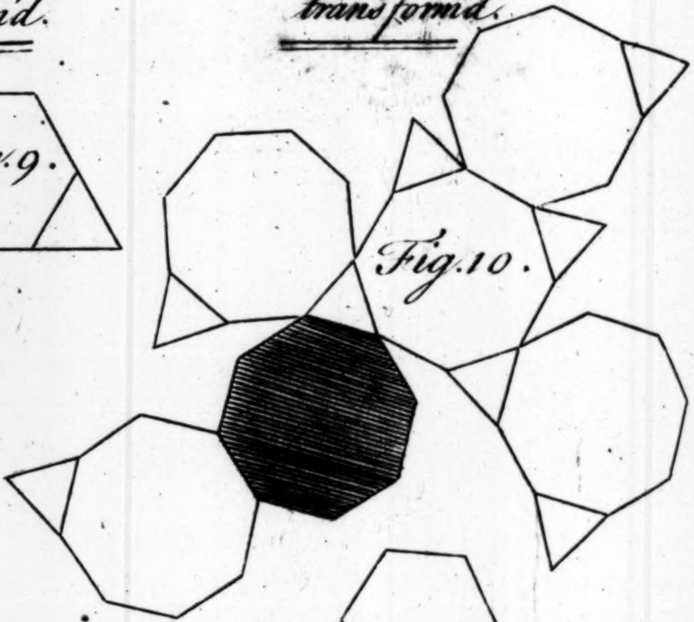


Fig. 10.

Dodecaedron,
transform'd.

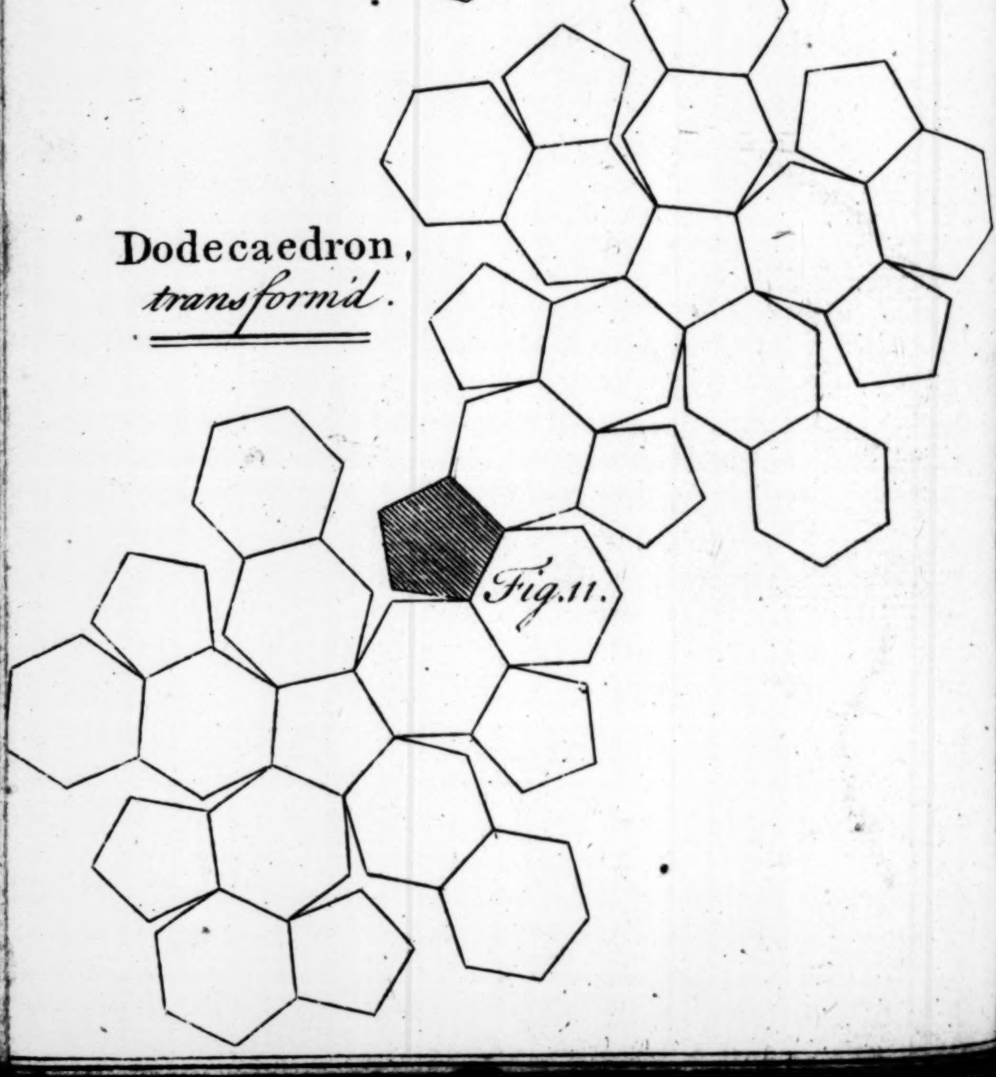
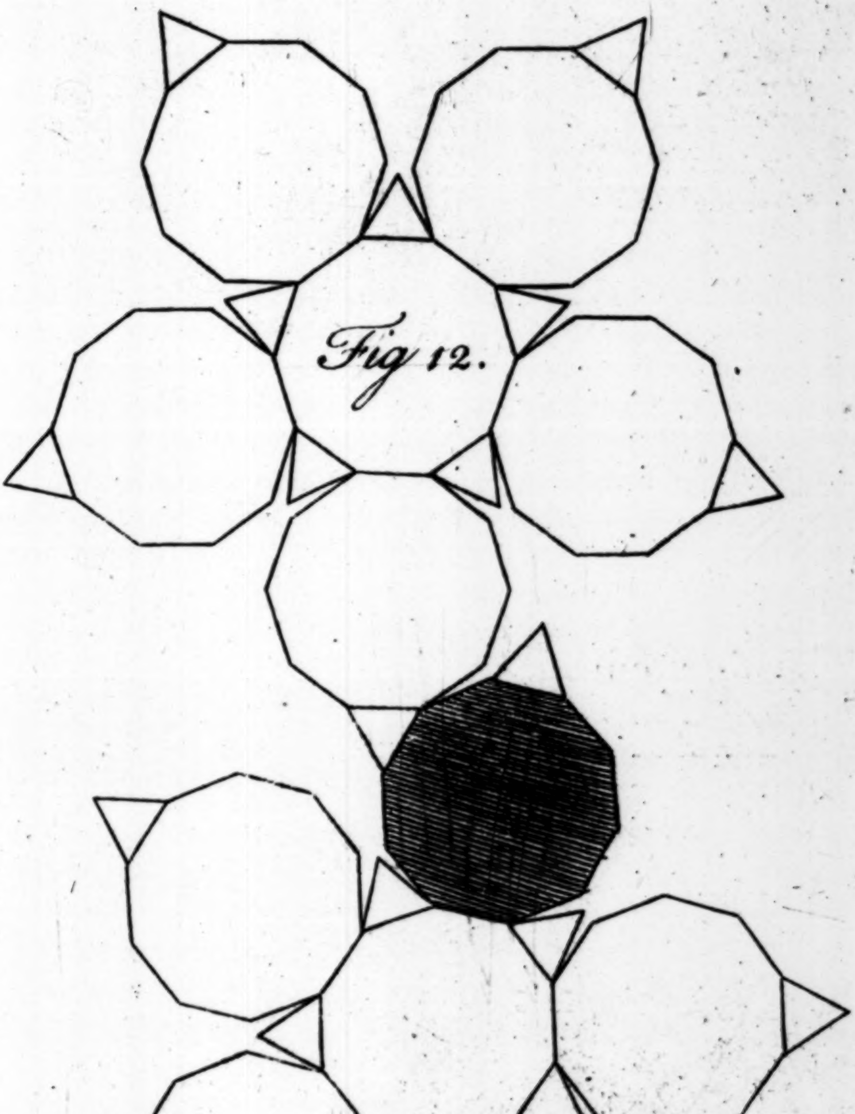


Fig. 11.

Icosaedron,
transform'd.



Tetraennaédron

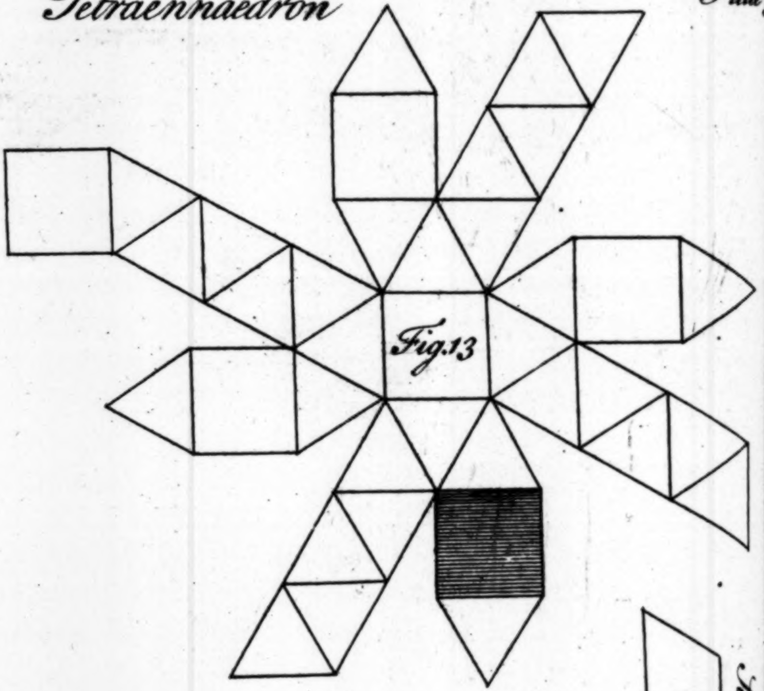


Fig. 13

Solid Rhombus

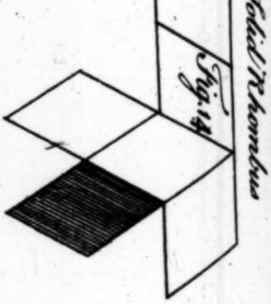


Fig. 14

Solid Rhomboides

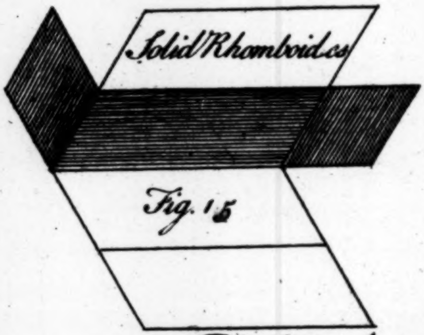


Fig. 15

Dodecarhombus

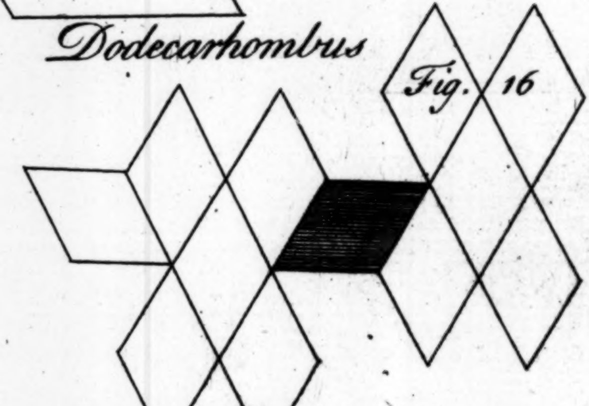


Fig. 16

Triexoctoedron

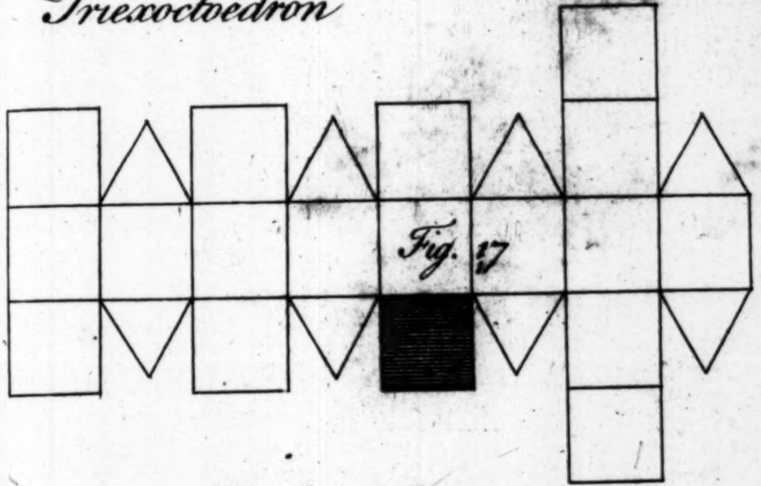


Fig. 17

Icosihexaedron

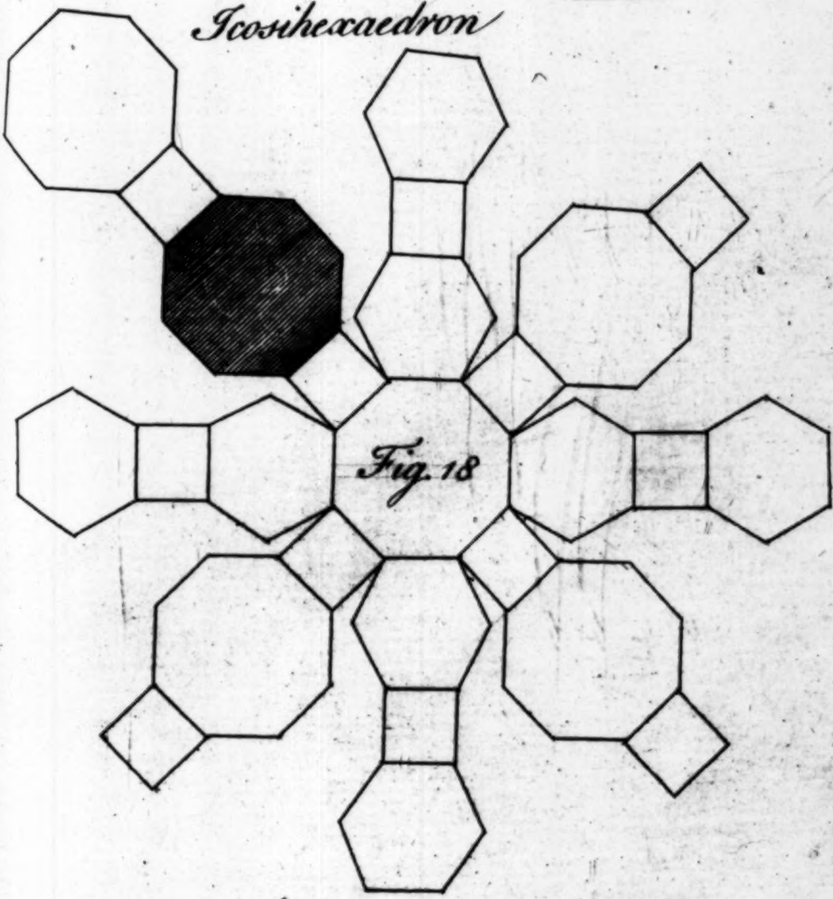
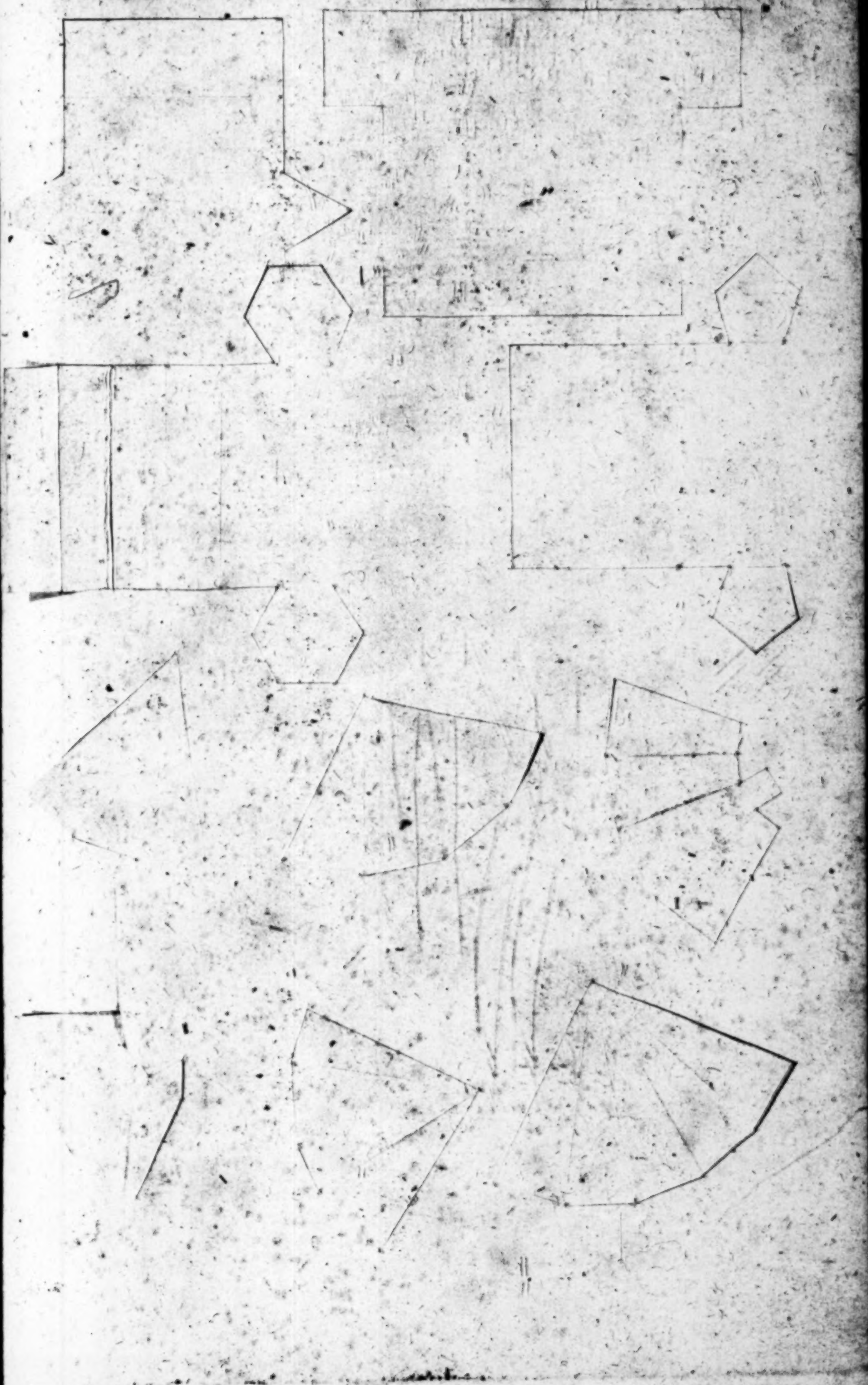
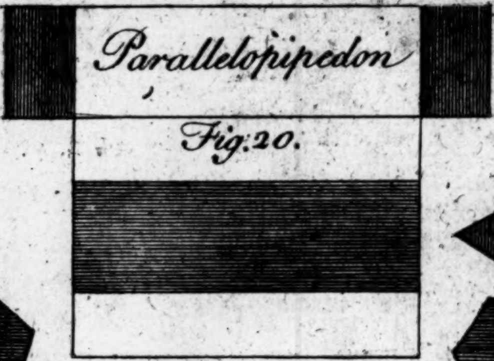


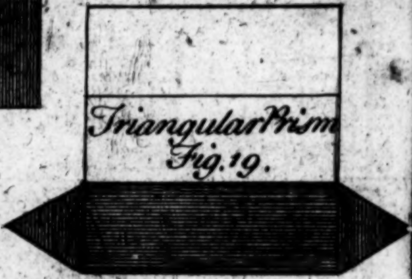
Fig. 18





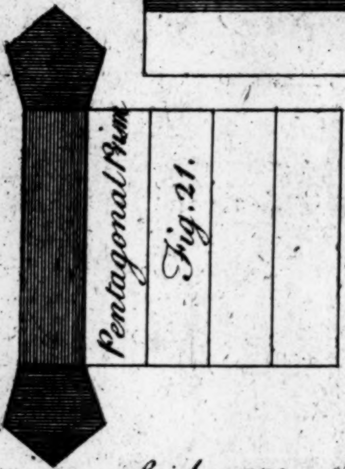
Parallelopipedon

Fig. 20.



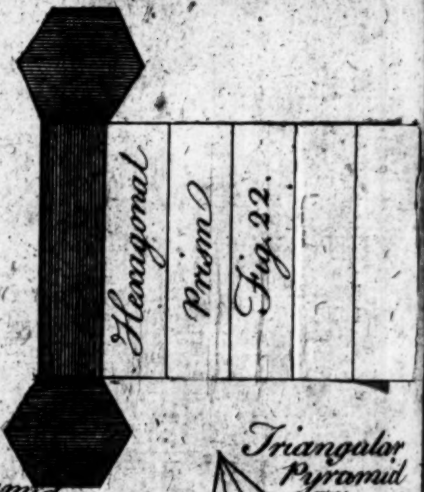
Triangular Prism

Fig. 19.



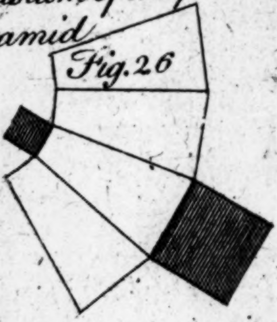
Pentagonal Prism

Fig. 21.



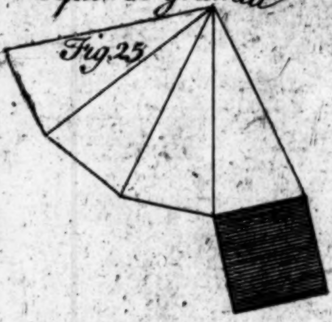
Hexagonal Prism

Fig. 22.



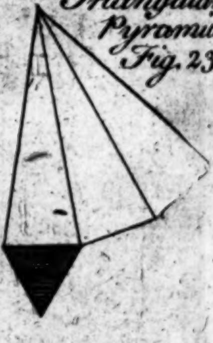
Frustum of a Square Pyramid

Fig. 26



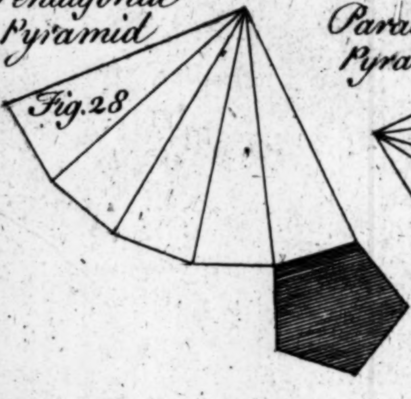
Square Pyramid

Fig. 25



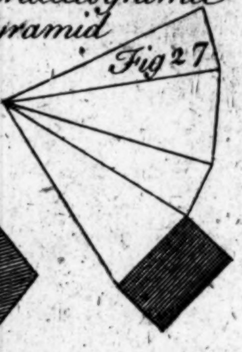
Triangalar Pyramid

Fig. 23



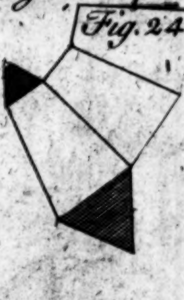
Pentagonal Pyramid

Fig. 28



Parallelogramic Pyramid

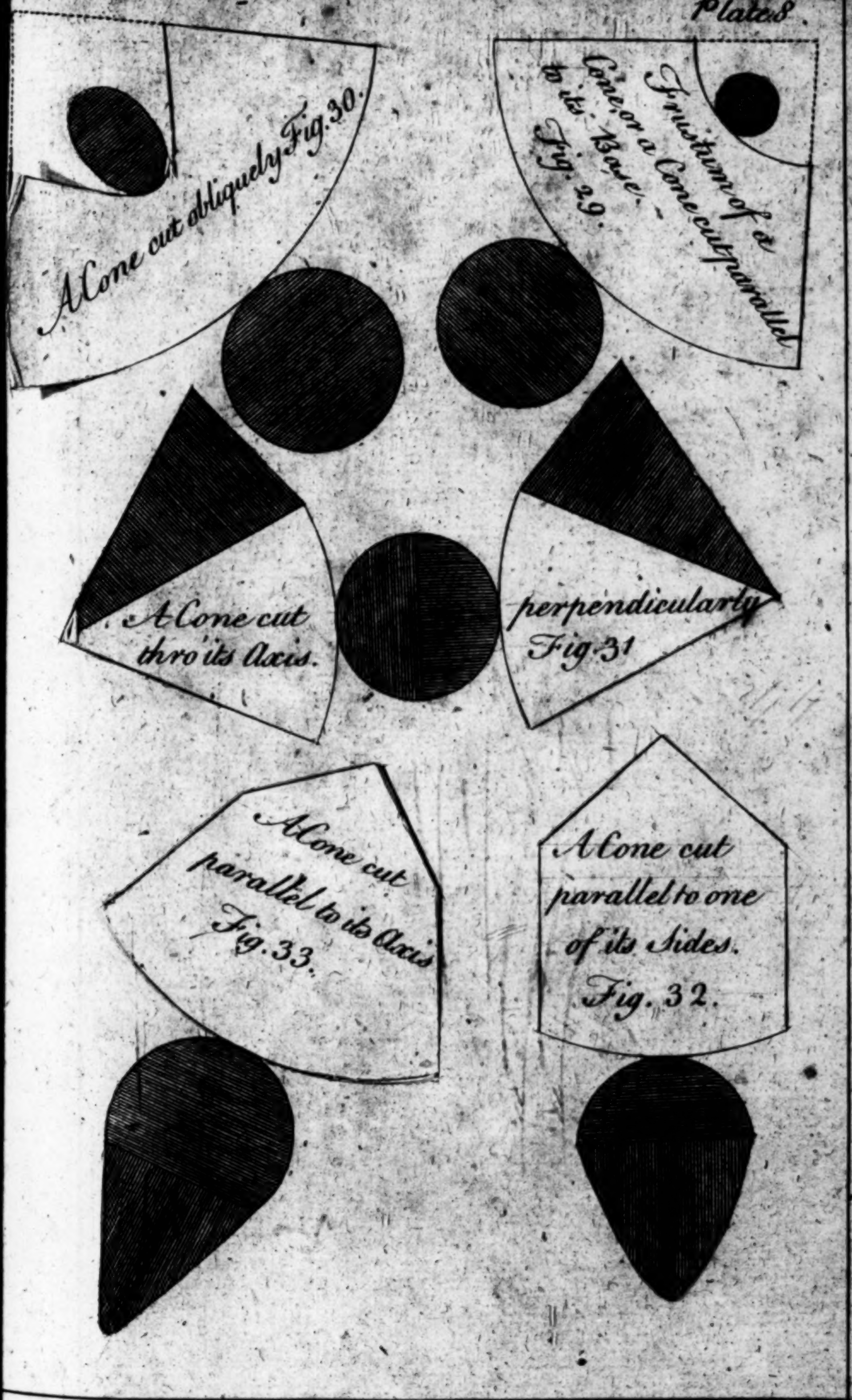
Fig. 27



Frustum of a Triangular Pyramid

Fig. 24.

Plate 8.



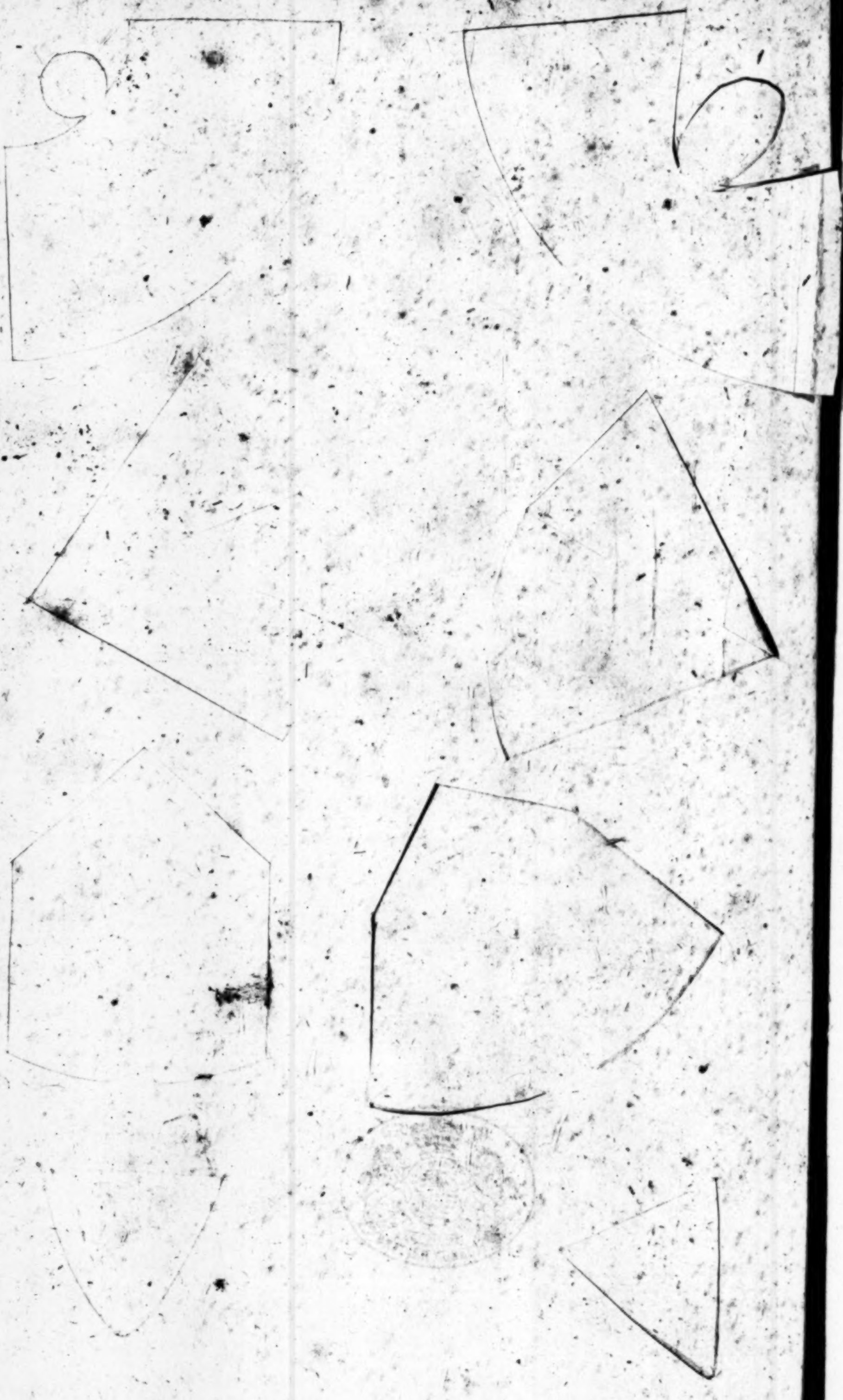
A Cone cut obliquely Fig. 30.

Frustum of a Cone, or a Cone cut parallel to its Base Fig. 29.

A Cone cut thro its Axis. perpendicularly Fig. 31

A Cone cut parallel to its Axis Fig. 33.

A Cone cut parallel to one of its Sides. Fig. 32.



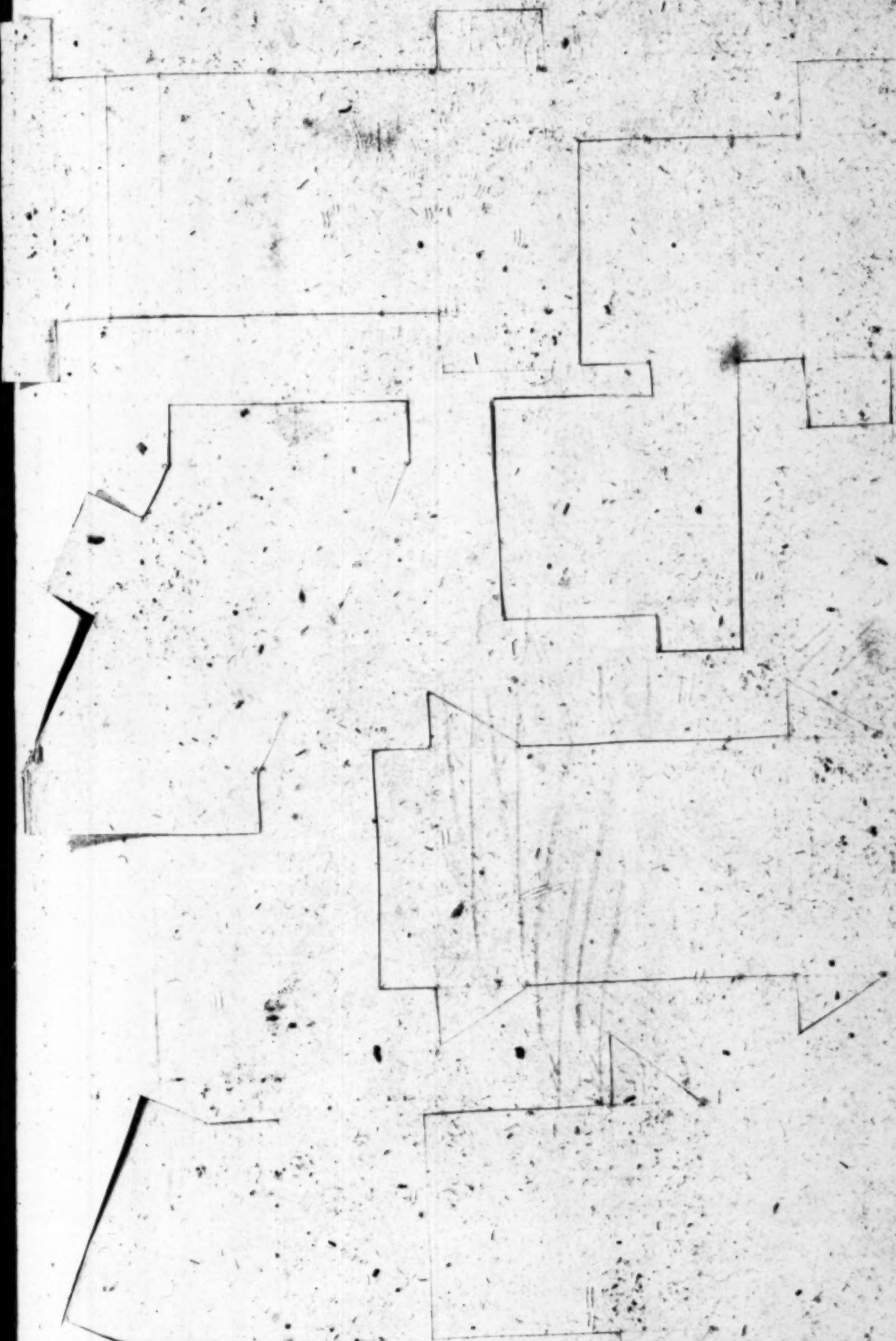


Plate 9

Euclid. 11 Prop. 25th

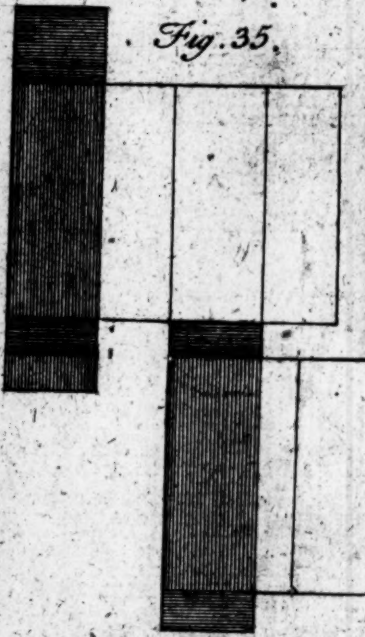


Fig. 35.

Euclid. 11. Prop. 25th

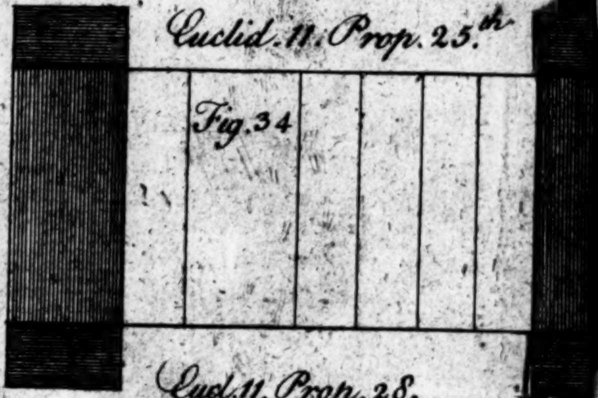


Fig. 34

Eucl. 11. Prop. 28.

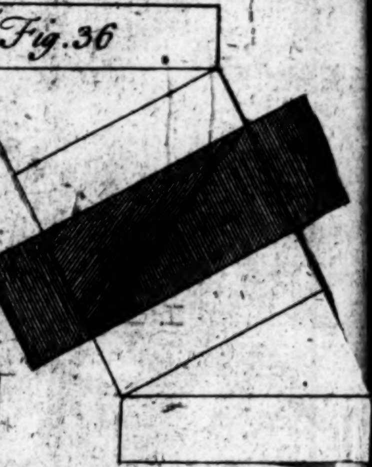


Fig. 36

Eucl. 11. Prop. 28

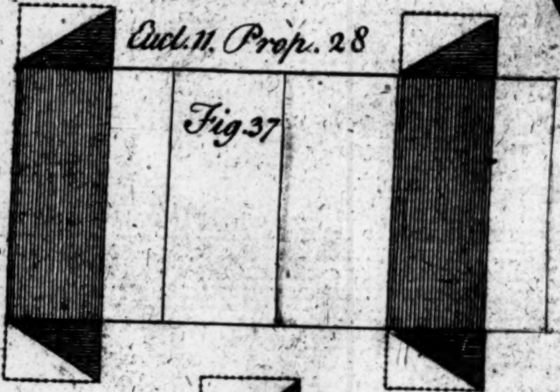


Fig 37

Eucl. 11. Prop. 40

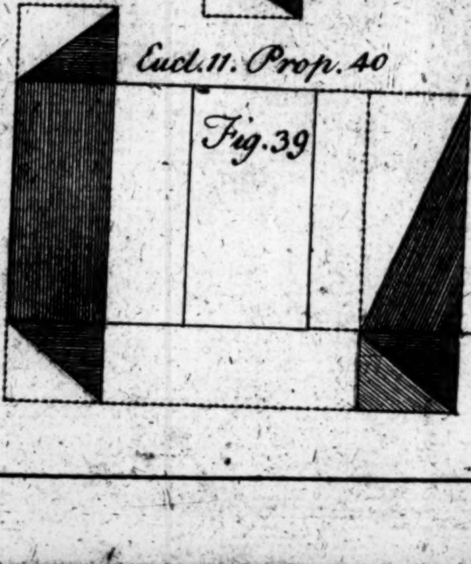


Fig. 39

Eucl. 11. Pro. 40

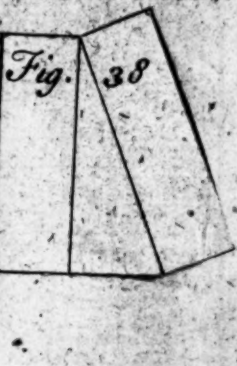
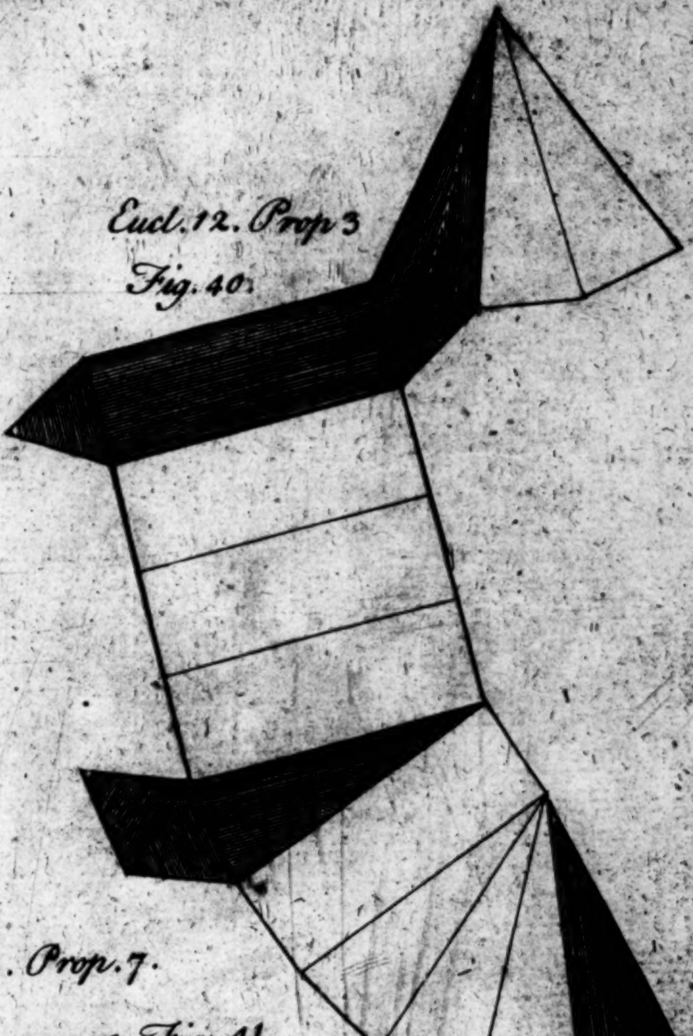


Fig. 38

Plate 10

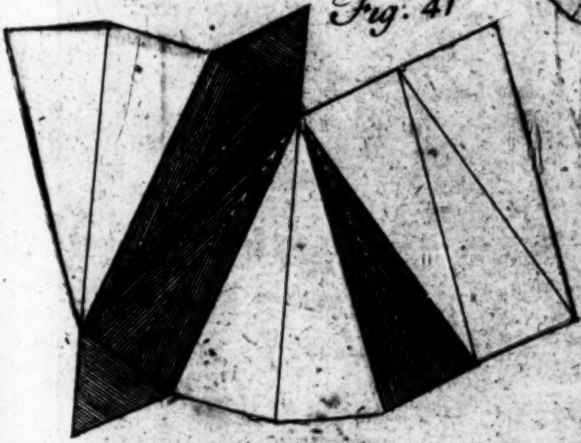
Eucl. 12. Prop 3

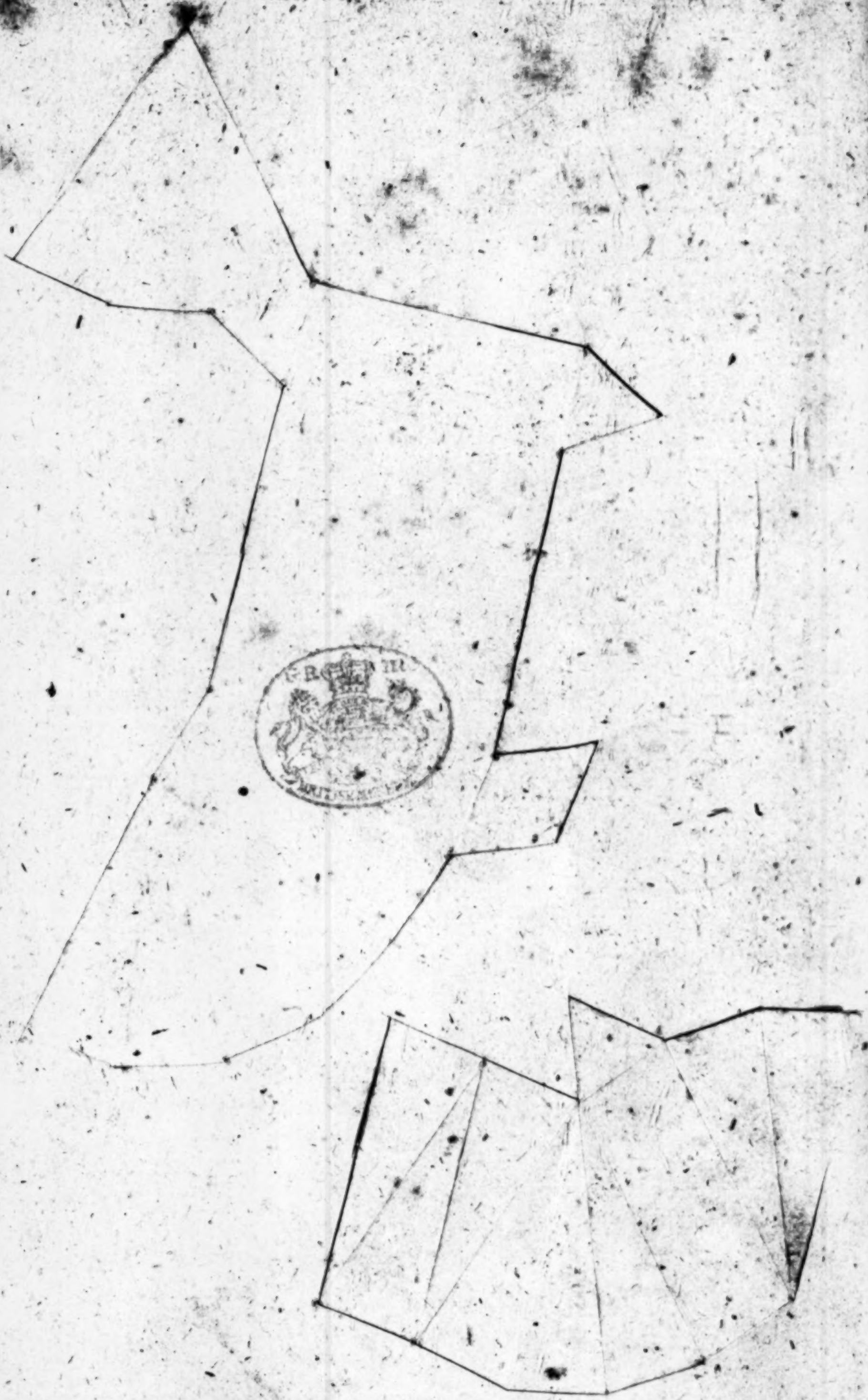
Fig. 40.



Eucl. 12. Prop. 7.

Fig. 41





G R III
BRITISH MUSEUM